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## Numerical Formulation on Crack Closing Effect In Buckling Analysis of Edge-Cracked Columns

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### ABSTRACT

In this paper, buckling of simply supported column with an edge crack is investigated numerically and analytically. Four different scenarios of damage severities are applied to a column, open crack assumption and the effect of closing crack in stability of the column which depends on position and size of cracks, are numerically compared. Crack surfaces contact is modeled with GAP element using SAP2000. For analytical solution, transfer matrix method, combined with fundamental solutions of the intact columns is used to obtain the capacity of slender prismatic columns. The stiffness of the cracked section is modeled by a massless rotational spring and governing equations are obtained explicitly for simply supported column from second-order determinant. As expected results show that the effect of a closing crack in presence of compressive load may lead to an increment in buckling load depending on crack depth and position. For the first time, a dimensionless formulation based on numerical results is presented in this study. Proposed formula predicts increment effect of closing crack in buckling results of a notched column.

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## 1. Introduction

One of the primary requirements imposed on structures is that they possess both stability and strength. Stability is the ability of structure to withstand the action of forces attempting to drive it out of a state of equilibrium. In classical stability analysis,

an elastic column is said to be stable if for any arbitrarily small displacement from its equilibrium position to the column either returns to its original undisturbed position or acquires an adjoined stable position when left to itself [1]. Buckling is one of the fundamental forms of instability of column structures. In practice, buckling is

characterized by a sudden failure of a structure member subjected to high compressive force.

Columns and compression members may contain various imperfections such as cracks. Mechanical vibrations, long term services or applied cyclic loads may results in the initiation of structural defects such as cracks in the structures. Therefore the determinations of the effect of this defect on the stability of structure is worthy of attention. Cracks lower the structural integrity and should be considered in the stability analysis of cracked structures [2]. Analytical research on buckling of circular rings and columns with cracks have been reported by Dimarogonas [3] using perturbation method, Leibowitz et al [4] and Leibowitz and Claus [5] used sine and cosine functions to model buckling loads for pre-cracked columns. Their model was approximate as it disregarded the discontinuity of the sine and cosine functions in their study. An analytical solution for buckling of slender prismatic columns with a single edge crack under concentric vertical loads has been proposed by Gurel and Kisa [6]. This method is based on rotational discontinuity of a cracked column at the crack location. Numerical studies employing the finite element method to study pre-cracked structures have been reported by Papadopolus [7], Chondros and Dimarogonas [8].

However, to the best of knowledge of present authors, open crack assumption is used for typical analytical methods to obtain critical buckling load of a notched column. In this study, using the transfer matrix method and fundamental solution of an intact column, a buckling analysis of slender prismatic columns of rectangular cross

section, with single nonpropagating edge crack is performed. The crack section is replaced with a massless rotational spring. FE and analytical methods are applied to analyze a simply supported pre-cracked column. FE analysis and crack closing modeling using GAP elements has been performed using SAP2000 package. New formula is proposed for the first time in this paper to determine crack closing effect in typical buckling analysis which is based on open crack assumption.

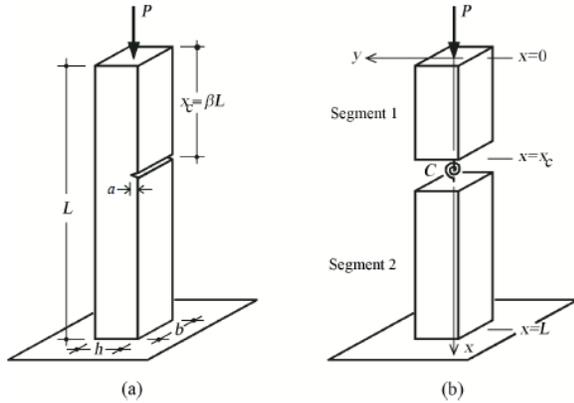
## 2. Formulation of the Problem

The formulation and figures presented here is a brief solution process of what Gurel and Kisa have reported in [6]. A column with a rectangular cross section and having a non-propagating edge crack is shown in figure1 (a). Typical analytical solutions consider the effect of crack with a massless rotational spring with flexibility  $C$ . It should be noted; this quantity is a function of the crack depth and height of the cross section of the column and can be written as [9]:

$$C = 5.346hf(\xi) \quad (1)$$

Where  $h$  is the height of the column and  $\xi = a/h$ , where  $a$  is the depth of the crack, as seen in figure 1(a).  $f(\xi)$  is called the local flexibility function and is given by [9]:

$$f(\xi) = 1.8624\xi^2 - 3.95\xi^3 + 16.375\xi^4 - 37.226\xi^5 + 76.81\xi^6 - 126.9\xi^7 + 172\xi^8 - 143.97\xi^9 + 66.56\xi^{10} \quad (2)$$



**Fig. 1.** (a) Column with a nonpropagating edge crack, (b) Its mathematical model [6].

As it can be seen in figure 1(b), column is divided by the rotational spring into 2 segments. The differential equation for buckling of segment 1 ( $0 \leq x \leq x_c$ ) can be written as [10]:

$$\frac{d^4 y_1}{dx^4} + k^2 \frac{d^2 y_1}{dx^2} = 0 \tag{3}$$

Where  $k^2 = P/EI$ , and  $P$  and  $EI$  are the axial compressive force and flexural rigidity, respectively.

In this case, the relationships among the displacement, slope, bending moment and shear force are

$$\begin{cases} \theta_1(x) = \frac{dy_1}{dx} \\ M_1(x) = -EI \frac{d^2 y_1}{dx^2} \\ V_1(x) = \frac{dM_1}{dx} - P \frac{dy_1}{dx} \end{cases} \tag{4}$$

The general solution of Eq. (3) is given by:

$$y_1(x) = A_1 + A_2 x + A_3 \sin(kx) + A_4 \cos(kx) \tag{5}$$

Using Eqs. (4) and (5), the following relationship can be written :

$$\begin{Bmatrix} y_1(x) \\ \theta_1(x) \\ M_1(x) \\ V_1(x) \end{Bmatrix} = B(x) \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \tag{6}$$

Where

$$B(x) = \begin{bmatrix} 1 & x & \sin(kx) & \cos(kx) \\ 0 & 1 & k \cos(kx) & -k \sin(kx) \\ 0 & 0 & P \sin(kx) & P \cos(kx) \\ 0 & -P & 0 & 0 \end{bmatrix} \tag{7}$$

The relationship between the parameters written above at the 2 ends of segment 1 can be expressed as:

$$\begin{Bmatrix} y_1(x_c) \\ \theta_1(x_c) \\ M_1(x_c) \\ V_1(x_c) \end{Bmatrix} = T_1 \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \tag{8}$$

In which

$$T_1 = B(x_c) B(0)^{-1} \tag{9}$$

[T1] transfers the parameters at the upper end ( $x=0$ ) to those at the lower end ( $x=x_c$ ) of segment 1 and is called t transfer matrix. The boundary conditions at  $x=x_c$  due the continuity among the displacements, bending moments and shear forces are:

$$\begin{cases} y_1(x_c) = y_2(x_c) \\ y_1''(x_c) = y_2''(x_c) \\ y_1'''(x_c) = y_2'''(x_c) \\ \Delta\theta(x_c) = Cy_1''(x_c) \end{cases} \tag{10}$$

Equation (10) can be written in matrix form as

$$\begin{Bmatrix} y_2(x_c) \\ \theta_2(x_c) \\ M_2(x_c) \\ V_2(x_c) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-C}{EI} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1(x_c) \\ \theta_1(x_c) \\ M_1(x_c) \\ V_1(x_c) \end{Bmatrix} \quad (11)$$

Substitution of Eq. (8) into Eq. (11) yields

$$\begin{Bmatrix} y_2(x_c) \\ \theta_2(x_c) \\ M_2(x_c) \\ V_2(x_c) \end{Bmatrix} = T_{1c} \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \quad (12)$$

In which

$$T_{1c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-C}{EI} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (13)$$

The equation for segment 2 can be obtained by using Eq. (12) and (8)

$$\begin{Bmatrix} y_2(0) \\ \theta_2(0) \\ M_2(0) \\ V_2(0) \end{Bmatrix} = T_{4 \times 4} \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \quad (14)$$

The matrix [T] has the following form:

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \quad (15)$$

The eigenvalue equations can be established by using Eq. (15) and the end conditions. For nontrivial solution, by setting the determinant of the matrix in Eq. (15) equal to zero, one obtains for pinned-pinned column:

$$T_{12}T_{34} - T_{14}T_{32} = 0 \quad (16)$$

After determining the elements of  $T_{ij}$  of the matrix [T] and then using the Eq. (16), the eigenvalue equations are obtained in explicit form:

$$\sin(kL) - Ck \sin(\beta kL) \sin[(1-\beta)kL] = 0 \quad (17)$$

Where  $\beta = xc/L$ , and L is length of column. By using MATLAB, a root-finder code is programmed to obtain roots (eigenvalues) of the above transcendental equation. The unknown parameter 'k' has been stated earlier.

### 3. Finite Element Analysis

Consider a prismatic column with rectangular section, material and geometry definition is presented in Table (1).

**Table 1.** Geometric and material properties of column

Element type	3D
Geometry type	Plane stress-Solid
Material	Isotropic
Width	40mm
Depth	20mm
Span	1m
Boundary condition	Simply supported
Poisson's Ratio	0.3
Mass density	7850 kg/m <sup>3</sup>
Modulus of elasticity	200 GPa

Column is modeled with eight-node solid element based on isoparametric formulation with incompatible modes and buckling analysis is performed to obtain corresponding critical loads for fundamental buckling load.

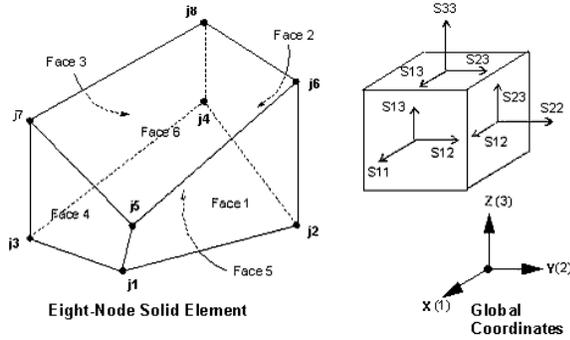


Fig 2. 8-node solid element stresses [11].

Crack contact is modeled with GAP element (Compression only -spring), which comes into function by defining real separation between GAP non-linear link and supported joints [11]. In This paper, crack width (separation between crack edges) is considered 1mm, which showed more agreement with Analytical solution results.

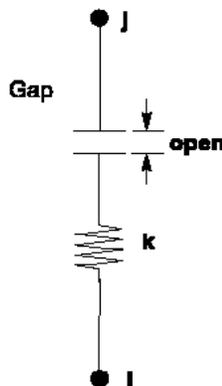


Fig 3. GAP model behavior [11].

### 3. 1 FE Validation

In first step, FE validation and mesh dimension analysis for first buckling load (PEuler) of an intact column has been analytically calculated. Fine meshing is applied to FE model to obtain more accuracy, maximum error between

$PE = \pi^2 EI / (KL)^2$  and finite element results is less than 8.6% which makes FE model acceptable.

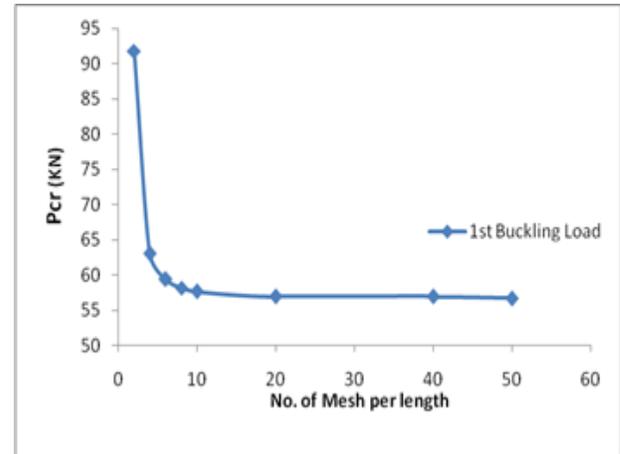
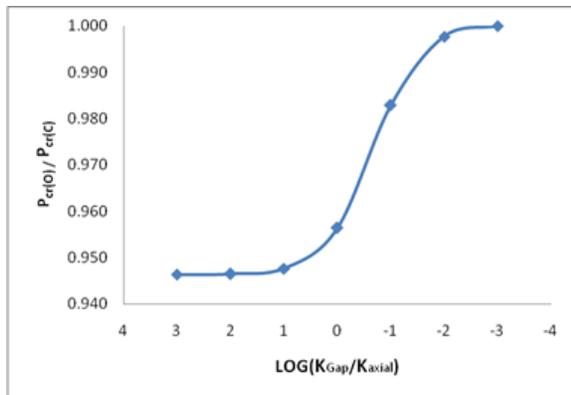


Fig 4. Buckling load variation with mesh dimension

Since comparison of results with this error, particularly for the cracked column creates difficulties to interpretation, appropriate crack width is considered to obtain more agreement between numerical and analytical results.

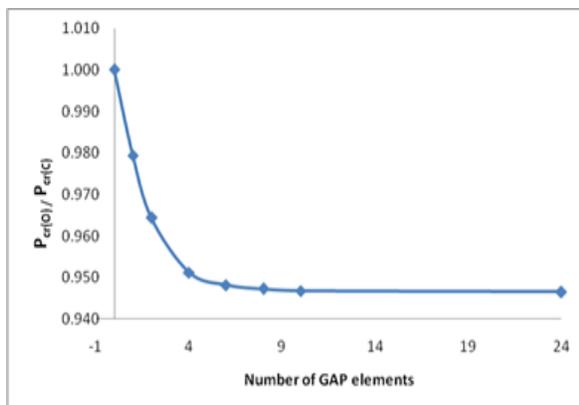
### 3. 2 Sensitivity analysis of Gap elements

For buckling analysis, two states for crack are considered: i) crack is open, ii) crack is closed partially in presence of compressive axial force. Crack closing is modeled with the aid of nonlinear GAP elements. Sensitivity analysis of numbers of GAP and stiffness is applied via buckling load ratio of cracked column. GAP stiffness converges for  $K_{Gap} = 1000 K_{axial}$ , which is shown in figure (5). Pcr(C) and Pcr(O) are buckling loads with GAP and without GAP, respectively.  $K_{axial}$  is axial stiffness of beam.



**Fig 5.** Sensitivity analysis of GAP stiffness

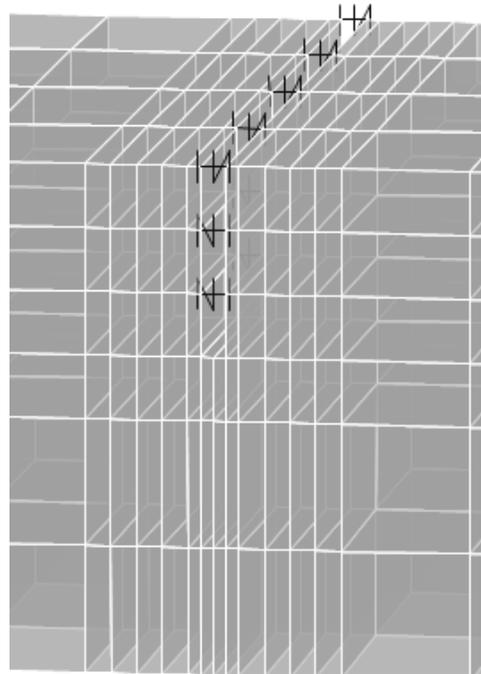
Figure (6) represents sensitivity analysis for number of GAP elements. While increasing number of GAP elements leads to convergence of buckling load ratio, location of GAP elements in crack is an important factor for rapid convergence.



**Fig 6.** Sensitivity analysis on No. of GAP elements

Final layout of GAPs in crack area is shown in figure (7). This layout will remain

constant during buckling analysis of cracked beam in closing crack mode.



**Fig. 7.** GAP layout in crack

### 3. Results

Four different scenarios,  $\xi=0.125, 0.25, 0.375$  and  $0.5$  for damage severity and crack location  $\beta=0.1, 0.25, 0.5$  due model symmetry are considered. Buckling analysis results are presented in Tables (2) for notched column.

**Table 2.** 1<sup>st</sup> Buckling load for cracked column  
1<sup>st</sup> Damage Scenario-( $\xi=0.125$ )

$P_{cr}$ (KN)	Analytical	FE-Open Crack	FE-Closing Crack	Error (%)
$\beta=0.5$	52.39	52.00	52.18	0.35
$\beta=0.25$	52.52	52.17	52.33	0.31
$\beta=0.1$	52.64	52.30	52.32	0.04

2<sup>nd</sup> Damage Scenario-( $\xi=0.25$ )

$P_{cr}$ (KN)	Analytical	FE-Open Crack	FE-Closing Crack	Error (%)
$\beta=0.5$	51.60	51.12	51.83	1.38
$\beta=0.25$	52.13	51.82	52.08	0.50
$\beta=0.1$	52.56	52.23	52.29	0.12

3<sup>rd</sup> Damage Scenario-( $\xi=0.375$ )

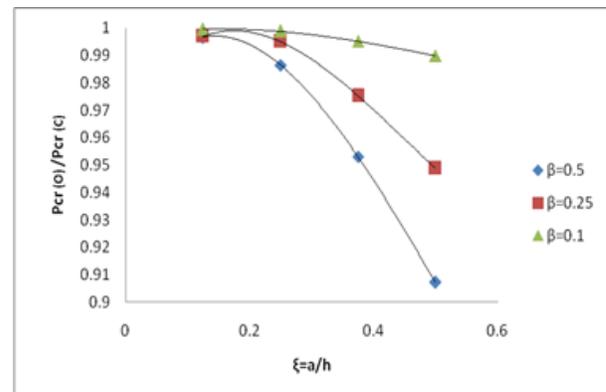
$P_{cr}$ (KN)	Analytical	FE-Open Crack	FE-Closing Crack	Error (%)
$\beta=0.5$	50.14	49.13	51.55	4.93
$\beta=0.25$	51.35	50.66	51.94	2.53
$\beta=0.1$	52.41	52.01	52.26	0.48

4<sup>th</sup> Damage Scenario-( $\xi=0.5$ )

$P_{cr}$ (KN)	Analytical	FE-Open Crack	FE-Closing Crack	Error (%)
$\beta=0.5$	47.50	46.20	50.93	10.23
$\beta=0.25$	49.87	48.96	51.59	5.38
$\beta=0.1$	52.11	51.62	52.15	1.02

For first buckling load, average error between analytical and finite element model (open crack) is less than 0.5 %. Errors which are reported in table (2) are comparison of open crack via close crack assumption in FE analysis results.

General trend of buckling load for first mode is descending when crack location approached to supports, where bending moments come to its minimum. It's evident from table (2), when the crack depth increases, the buckling load decrease as expected. Maximum difference for first critical buckling load occurs in  $\beta=0.5$ . The crack location affects buckling results depending on mode number.



**Fig. 8.** Buckling analysis ratio for Open/Closing crack assumption

Proposed formulation presents an equation based on crack depth and position. Mean squared error for all equations is  $R^2=1$ . formulation is given

$$\frac{P_{cr(O)}}{P_{cr(C)}} = \Pi \xi, \beta = a\xi^3 + b\xi^2 + c\xi + d \quad (18)$$

$$\begin{cases} a = -14.18\beta^2 + 10.56\beta - 0.817 \\ b = 16.84\beta^2 - 13.28\beta + 0.996 \\ c = -5.28\beta^2 + 3.948\beta - 0.298 \\ d = 0.453\beta^2 - 0.332\beta + 1.024 \end{cases} \quad (19)$$

#### 4. Conclusions

Buckling analysis of simply supported column using analytical and finite element methods has been performed to investigate crack closing effect and following conclusions are drawn as follows:

Open crack assumption in buckling analysis represents more conservative results. Considering crack closing increase buckling load depending on crack location and damage severities. There was very good agreement between numerical and analytical results for buckling analysis (maximum discrepancy less than 0.5%). As expected, the load carrying capacity decreases as the crack depth increases. When crack is located in section of maximum bending in each mode of buckling, open crack assumption cause more loss of strain energy and consequently large difference of results. In first critical buckling load which is more of interest, maximum difference between open-close crack assumption for  $\xi < 0.5$  which are developed in earlier states of damage is about 10%, that shows open crack assumption is not out of admissibility. Numerical formula which is a function of crack parameters is derived with best polynomial regression. This paper aimed to

formulate discrepancy of open and closing crack assumption in buckling analysis results for earlier damage states.

In addition, buckling happens in the direction of minimum moment of inertia. When buckling plane is perpendicular to the crack direction (in plane), it is realistic to assume that cracks are open in deformed shape. In this way, buckling plane is ensured to be in plane of crack direction for which closing crack occurred.

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