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ARTICLE INFO

Article history:
Received: 15 April 2019
Accepted: 10 October 2019

Keywords:
Damage Detection, Rayleigh-Ritz, Mode Shapes, Natural Frequency.

ABSTRACT

As a result of environmental and accidental actions, damage occurs in structures. The early detection of any defect can be achieved by regular inspection and condition assessment. In this way, the safety and reliability of structures can be increased. This paper is devoted to propose a new and effective method for detecting, locating, and quantifying beam-like structures. This method is based on Rayleigh-Ritz approach and requires a few numbers of natural frequencies and mode shapes associated with the undamaged and damaged states of the structure. The great advantage of the proposed approach against the other methods is that it considers all kinds of boundary and damping effects. To detect damage using the penalty method, this article considers lumped rotational and translational springs for determining the boundary conditions. Result will show that the proposed method is an effective and reliable tool for damage detection, localization, and quantification in the beam-like structures with different boundary conditions even when the modal data are contaminated by noise.

1. Introduction

In civil, mechanical, and aerospace engineering, damage diagnosis of structures has received considerable attention of researchers. Damage detection methods provide the continuous inspection of the safety and integrity of structures. Four main levels of these techniques are detection, localization, severity investigation and also structure remained life estimation [1]. Up to now, a large amount of damage detection strategies have been developed. Most of them are based on finite element method (FEM)
It is worth emphasizing that the structural responses and modal parameters are the functions of mechanical properties of the structure. Hence, any changes in physical characteristics of a structure will alter the responses. Accordingly, the damage diagnosis schemes based on FEM have been established. These techniques update the FEM model to match the analytical and measured data. In this way, mechanical parameters of the damaged structure can be estimated. These techniques are divided into two groups including the static and dynamic approaches. In the latter one, the modal parameters such as the natural frequencies and mode shapes are used, and the responses such as displacements and strains are measured in static approaches [3, 4]. It should be mentioned that some of the damage identification techniques take advantage of both static and dynamic data. In the following, the well-known damage detection methods of beams are reviewed briefly.

In 1996, Stubbs and Kim [5] found the damage location without using the baseline modal parameter in a two-span concrete beam by utilizing FEM. With the help of dynamic stiffness, Maeck et al. [6] proposed two strategies for detecting stiffness degradation in reinforced concrete beams. They validated their method by performing experiments. In another research work, Maeck and De Roeck [7] assessed gradually damaged reinforced concrete beams by calculating the modal bending moments and curvatures. Moreover, Kokot and Zembaty [8] measured both translational and rotational degrees of freedom for damage detection of beams. In this research, a minimization problem was solved. By using the global fitting technique, Myung-Keun et al. [9] located damage in beams. Note that this algorithm was the extension of the gapped fitting method. By employing time modal features and artificial neural networks, Park et al. [10] presented a sequential damage detection method for beams. Their approach included two phases consisting of time-domain and modal-domain damage monitoring. Sung et al. [11] deployed modal flexibility for damage detection of cantilever beam-type structures. Additionally, Eraky [12] used a damage index algorithm as a tool for localizing damages occurred in flexural structural elements. This index was based on modal strain energy. Then, they experimentally verified the mentioned algorithm. Limongelli [13] carried out experiments on a post tensioned concrete beam for assessing the possibility to detect early warning signs of deterioration based on static and/or dynamic tests. In most of the damage detection methods, FEM was employed. Although the FEM scheme is has extensively been utilized in damage detection of structures, only few approaches are developed based on the well-known Rayleigh-Ritz technique. These strategies are briefly reviewed below.

Deobald and Gibson [14] utilized the Rayleigh-Ritz algorithm for finding the elastic constants of the orthotropic plates with clamped and free edges. Cao and Zimmerman [15] deployed the Ritz vector to monitor the health of structures. In this process, the minimum rank perturbation theory was utilized. Also, these researchers concluded that the Ritz vectors are more sensitive to damage than the mode shapes.
Based on Rayleigh-Ritz approach, Li et al. [16] specified the location of damage in plate-like structures. Moreover, they suggested two new damage indices based on modal strain energy. Afterwards, Sohn and Law [17] applied load-dependent Ritz vectors in Bayesian probabilistic damage detection. In 2011, Sarker et al. [18] employed the Ritz method for damage detection of circular cylindrical shell. García et al. [19] proposed a new technique for damage localization in laminated composite plates by using mode shape derivatives. Moreover, they suggested another damage detection approach for localizing the damage in the above-mentioned plates. In this approach, they took advantage of the higher order derivatives, and in a similar manner, they utilized the Ritz technique. Maghsoodi et al. [20] presented a simple technique for detecting, locating and quantifying multiple cracks in Euler–Bernoulli multi-stepped beams. To model cracks, they used rotational springs. They observed that the location and severity of damage were related to the natural frequencies of the beam. In this work, global interpolation functions were utilized the same as Rayleigh–Ritz method, instead of applying piecewise continuous mode shapes for a multi-step beam. Gharighoran et al. [21] deployed the Ritz strategy for damage detection of reinforced and post-tensioned concrete beams. This technique was only applicable to simply supported beams.

This paper aims to propose a simple and robust method based on Ritz scheme for locating and quantifying damage in beam-like structures with any kinds of boundary conditions. In fact, this work is an improvement on the technique proposed by Gharighoran et al. [21]. In this study, damage is modeled by reducing flexural stiffness. Moreover, the proposed strategy requires some of the natural frequencies and mode shapes of the healthy and damage structure.

2. Governing Equations

To have a review of natural frequencies and mode shapes, in general, the structural motion could not be harmonic. However, there are certain special initial displacements that will cause harmonic vibrations. These special initial deflections are called mode shapes, and the corresponding frequencies of vibration are called natural frequencies. Both of these parameters are related and could be captured by mass and stiffness matrices. The differential equation which governs the free vibration behavior of the beam has the subsequent equation [22]:

\[ M \ddot{Z} + C \dot{Z} + KZ = 0 \]  (1)

In which \( M, C \) and \( K \) denote mass, damping and stiffness matrices, respectively. Moreover, \( \ddot{Z}, \dot{Z}, \) and \( Z \) are the acceleration, velocity and displacement vectors, respectively. If the proportional damping is deployed, Eq. (1) can be rewritten as follows [22]:

\[ M \ddot{Z} + K_d \dot{Z} = 0 \]  (2)

\[ K_d = K + Hj; \quad j = \sqrt{-1} \]  (3)

Where \( H \) denotes the Rayleigh's damping matrix, which can be expressed as [22]:

\[ H = \alpha K + \beta M \]  (4)

In this expression, \( \alpha \) and \( \beta \) are constant values [22]. Accordingly, the following equations can be achieved for the damped and un-damped structures, respectively [21]:

\[ K\Psi_p = \omega^2_p M\Psi_p \quad p=1,\ldots,r \]  (5)
\( \mathbf{K}\Psi_p = \tilde{\omega}_p^2 \mathbf{M}\Psi_p \quad p=1, \ldots, r \)  \hspace{1cm} (6)

where the \( p^{th} \) natural frequencies of the damped and undamped structures are shown by \( \tilde{\omega}_p \) and \( \omega_p \), respectively. Furthermore, the \( p^{th} \) mode shapes of the damped and undamped structures are \( \Psi_p \) and \( \tilde{\Psi}_p \), respectively. Additionally, \( \mathbf{K}_d \) denotes the damped stiffness matrix. Besides, \( r \) is the number of mode shapes taken into account.

As a result of damage, the characteristics matrices of the structure are altered. Consequently, Eqs. (5) and (6) change into the following forms [21]:

\( \tilde{\mathbf{K}}\tilde{\Psi}_p = \tilde{\omega}_p^2 \tilde{\mathbf{M}}\tilde{\Psi}_p \quad p=1, \ldots, s \)  \hspace{1cm} (7)
\( \mathbf{K}_d \tilde{\Psi}_p = \tilde{\omega}_p^2 \tilde{\mathbf{M}}\tilde{\Psi}_p \quad p=1, \ldots, s \)  \hspace{1cm} (8)

where the \( p^{th} \) natural frequencies of the damped and undamped damaged structures are shown by \( \tilde{\omega}_p \) and \( \omega_p \), respectively. Furthermore, the \( p^{th} \) mode shapes of the damped and undamped structures are \( \tilde{\Psi}_p \) and \( \Psi_p \), respectively. Additionally, \( \tilde{\mathbf{K}}_d \) and \( \tilde{\mathbf{K}} \) denote the damped and undamped stiffness matrix of the damaged structure, respectively. Also, \( \tilde{\mathbf{M}} \) is the mass matrix of the damaged structure. Besides, \( p \) is the number of available mode shapes of the damaged structure. Accordingly, the subsequent orthogonality conditions can be obtained[21]:

\( \Psi_p^T \tilde{\mathbf{M}} \Psi_q = \delta_{pq} \)  \hspace{1cm} (9)
\( \tilde{\Psi}_p^T \tilde{\mathbf{K}} \tilde{\Psi}_q = \tilde{\omega}_q^2 \delta_{pq} \)  \hspace{1cm} (10)
\( \tilde{\Psi}_p^T \mathbf{K}_d \tilde{\Psi}_q = \tilde{\omega}_q^2 \delta_{pq} \)  \hspace{1cm} (11)

where \( \delta \) is Kronecker delta operator. In usual, it is assumed that the mass matrix of the damaged and undamaged structure are similar [23]. As a consequence, only their stiffness matrices are different.

**3. Damage Index**

In this section, a local index is introduced for describing damage. The global behavior of the structure is dependent on this parameter. In other words, it is sensitive to structural changes induced by occurrence of damage. In finite element formulations, damage is generally represented by a reduction in the stiffness of the individual finite elements, and it is presumed that the elemental stiffness matrix decreases uniformly due to damage. Therefore, the damage identification is conducted at the element level. The change of the stiffness of the \( e^{th} \) element can be defined as follows [21]:

\( k_{Se} = k_e - \hat{k}_e \)  \hspace{1cm} (11)

In this relation, \( k_e \) and \( \hat{k}_e \) are the stiffness matrices of the \( e^{th} \) element before and after the occurrence of damage, respectively. The stiffness matrix of the damaged element is given by [21]:

\( \hat{k}_e = (1 - \alpha_{Se})k_e \)  \hspace{1cm} (12)

where \( \alpha_{Se} \) denotes the sound index of the element. This parameter ranges from 0 to 1. If the \( e^{th} \) element is completely destroyed, \( \alpha_{Se} \) is equal to 1. On the other hand, \( \alpha_{Se} \) is zero when it is healthy. Consequently, the damage index can be expressed as the following form:

\( \alpha_{De} = 1 - \alpha_{Se} \)  \hspace{1cm} (13)

Therefore, the damage index of a beam element is written as [21]:

\( \alpha_{De} = \frac{(\tilde{E}I)_e}{(EI)_e} \)  \hspace{1cm} (14)

In this equation, \( (\tilde{E}I)_e \) and \( (EI)_e \) are the flexural stiffness of the \( e^{th} \) element after and prior to damage, respectively. Accordingly, the global stiffness matrix of a damaged structure can be expressed as [21]:

\( \hat{\mathbf{K}} = \sum_{e=1}^{m} \alpha_{De} \mathbf{K}_e \)  \hspace{1cm} (15)

It should be mentioned that \( m \) is the number of elements.
4. Formulating the Damage Detection Problem Using the Ritz Method

The strain energy of the Euler-Bernoulli beam depicted in Fig. 1, can be computed as follows [24]:

\[
\Pi = \frac{1}{2} \int_0^L EI(x) \left( \frac{d^2 w(x)}{dx^2} \right)^2 dx
\]  
(16)

where the flexural stiffness and vertical displacement functions are denoted by \( EI(x) \) and \( w(x) \), respectively.

![Fig. 1. The finite element model of a beam.](image)

In Eq. (16) and Fig. 1, \( L \) is the beam length. Herein, for imposing the boundary conditions, translational and rotational springs are deployed. As a consequence, Eq. (16) is rewritten as below [24]:

\[
\Pi = \frac{1}{2} \int_0^L EI(x) \left( \frac{d^2 w(x)}{dx^2} \right)^2 dx + \sum_{i}^{nt} \frac{1}{2} K_i^t w^2(x_i) + \sum_{i}^{nr} \frac{1}{2} K_i^r w^2(x_i)
\]  
(17)

In this equation, \( K_i^t \) and \( K_i^r \) are the stiffness of the \( i^{th} \) translational and rotational springs, respectively. Note that the definition of these kinds of springs are available in [24]. In addition, \( nt \) and \( nr \) denote the number of translational and rotational springs. Moreover, the location of the \( i^{th} \) spring is shown by \( x_i \). To estimate the response of the system, the following functions are selected [24]:

\[
\Pi = \frac{1}{2} \int_0^L EI(x) \left( \frac{d^2 w(x)}{dx^2} \right)^2 dx + \sum_{i}^{nt} \frac{1}{2} K_i^t w^2(x_i) + \sum_{i}^{nr} \frac{1}{2} K_i^r w^2(x_i)
\]  
(18)

It should be highlighted that the boundary conditions are automatically satisfied by adjusting the stiffness of the aforesaid springs. These functions are stored in a shape function vector as the coming shape:

\[
N_i(x) = \begin{cases} 
\left( \frac{x}{L} \right)^i & i = 1, 2, 3 \\
\sin \left( \frac{(i-3)\pi x}{L} \right) + \cos \left( \frac{(i-3)\pi x}{L} \right) & i = 4, 5, ..., n
\end{cases}
\]  
(19)

Note that, number of entries of this vector is \( n \). Consequently, the Ritz solution for the problem can be expressed as the next form[25]:

\[
w(x) = N(x)B
\]  
(20)
\[
K = \int_0^L EI(x) \left( \frac{d^2 N(x)}{dx^2} \right)^T \left( \frac{d^2 N(x)}{dx^2} \right) dx \tag{21}
\]
\[
M = \int_0^L \rho A(x) N(x)^T N(x) dx \tag{22}
\]

In these relations, the mass density and the cross-sectional area function of the beam are shown by \( \rho \) and \( A(x) \), respectively. For the damaged structure, the stiffness matrix has the succeeding appearance [21]:

\[
\hat{K} = \sum_{i=1}^m \int_0^1 (1 - \alpha_d) \left( \frac{d^2 N_i(\xi)}{d\xi^2} \right)^T \left( \frac{d^2 N_i(\xi)}{d\xi^2} \right) L \left[ J_i \left| d\xi \right. \right] \tag{25}
\]

\[
M = \sum_{i=1}^m \int_0^1 \rho A_{ei}(\xi) N(\xi)^T N(\xi) \left| J_i \right| d\xi \tag{26}
\]

In these relations, \( EI_{ei}(\xi) \) and \( A_{ei}(\xi) \) are the flexural stiffness and cross-sectional area of the \( i \)th element. Moreover, \( J_i \) denotes the Jacobian of transformation.

In an undamped and damped structure, natural frequencies and mode shapes can be obtained for damage scenarios with the help of relations (6) and (7), respectively. Note that; the eigen-values are the natural frequencies of the structure, but the eigenvectors includes the coefficients of Eq. (20).

By using the next equation, the mode shapes can be achieved [26]:

\[
\Psi_i = N(x)^T \psi_i \tag{27}
\]

in which \( \Psi_i \) is the eigen-vector.

### 5. Inverse Solution

Based on the above-mentioned equations, the inverse problem required for estimating the severity and location of damage can be established. In this formulation, the element damage indices can be expressed in natural coordinates as follows:

\[
\alpha_d = \alpha_d(\xi) \tag{24}
\]

where \( \xi \) is the natural coordinate corresponding to each element. Accordingly, the global stiffness and mass matrices can be rewritten as, respectively [26]:

\[
\hat{K} = \int_0^L (1 - \alpha_d) EI(x) \left( \frac{d^2 N(x)}{dx^2} \right)^T \left( \frac{d^2 N(x)}{dx^2} \right) dx \tag{23}
\]

Experimental modal data of the damping of structures are needed. It is obvious that the following equation can be written based on orthogonality conditions [22]:

\[
\hat{\Psi}^T \hat{K} \hat{\Psi} = \hat{\Psi}^T \hat{M} \hat{\Psi} \Omega^2 \tag{28}
\]

or,

\[
\hat{\Psi}^T \hat{K} \hat{\Psi} - \hat{\Psi}^T \hat{M} \hat{\Psi} \Omega^2 = 0 \tag{29}
\]

where \( \Psi \) is matrix, which stores the measured mode shapes, and \( \Omega^2 \) is a diagonal matrix including the square of the measured natural frequencies. Due to applying the Ritz method, the number of rows of these matrices are equal to the number of terms utilized in Ritz solution (\( n \)). In other words, their number of rows are equal to the number of unknowns which exist in the response solution. It should be pointed out that number of columns of these matrices equals to the number of measured mode shapes. Moreover, it needs to mention that \( \hat{K} \) is replaced by \( \hat{K}_d \) for damped structures. It is worthwhile to highlight that these matrices include the damage indices. Theoretically, the right side of Eq. (29) is a symmetric matrix, whose
entries are zero. However, in practice, it is not true due to the existence of measurement errors.

Accordingly, by extracting the unknown damage indices from the stiffness matrix, the following system of equations can be achieved [21]:

\[ A\alpha = R \] (30)

In this equality, vector \( \alpha \) contains unknown damage indices. Also, \( A \) and \( R \) are coefficient matrix and residual error vector, respectively. Note that; number of rows of \( R \) and \( A \) are \( s(s + 1)/2 \). It should be reminded that number of columns of matrix \( A \) is equal to \( s \). Therefore, this matrix may or may not be square. If number of the rows and columns are not equal, the pseudo-inverse is required for finding \( \alpha \) as follows[21]:

\[ \alpha = A^+ R \] (31)

Where

\[ \text{if } \frac{s(s+1)}{2} > s \Rightarrow A^+ = (A^T A)^{-1} A^T \] (32)

\[ \text{if } \frac{s(s+1)}{2} > s \Rightarrow A^+ = (A^T A)^{-1} A^T \] (33)

In these relations, the pseudo-inverse of the coefficient matrix is denoted by \( A^+ \).

6. Numerical Examples

In this section, the accuracy and efficiency of the proposed method is evaluated by performing several numerical analyses. Using a damage scenario in the first step, the mode shapes and natural frequencies of the structure are calculated by using the structural analysis. It should be note that the Ritz formulation introduced in section 4 is applied for this purpose. In other words, the structural analysis is employed instead of modal tests. By finding the mode shapes and natural frequencies, the authors' damage detection procedure begins. It should be recalled that only few number of mode shapes and natural frequencies are required in the suggested method for estimating the magnitude and location of damage. It is worthwhile to highlight that a special-purpose program is developed for calculating the mode shapes and natural frequencies based on section 4. Additionally, this program is able to perform damage detection with the help of the proposed method.

6.1. A Two-Span Beam

This section assesses a damaged two-span concrete beam with 0.2m×0.3m cross-section, 0.00045 moment of inertia (I), 25 GPa Young’s modulus (E), 2500 kg/m³ density (\( \rho \)), and 16 m length (L). For modeling the boundary conditions, the stiffness of the transitional spring (\( K_t \)) corresponds to EI×10⁵. It is presumed that the beam includes 16 subdivisions. Note that this beam was previously modeled in [27].

For the problem of damage detection, two patterns are considered by using the two types of stiffness reduction factors applied to some subdivisions. The amounts of stiffness reductions are identical to 5% and 10% for the first damage pattern and 10% and 20% for the second one. Figure 2 illustrates the two damage patterns as well as the subdivisions with the reduced stiffness. It is clear that the 16 elements of this beam are harmed in these damage patterns. Prior to beginning the damage detection process, the mode shapes and the first few natural frequencies of these two cases are required. As previously mentioned, the introduced formulation in section 4 is applied to achieve
this goal. It is noticeable that the six natural frequencies and mode shapes of the beam are calculated to use in the two damage scenarios.

![Fig. 2.a](image)

**Fig. 2.a)** First Damage scenario with depth of the stiffness losses ranges from 5% to 10%.

![Fig. 2.b](image)

**Fig. 2.b)** Second Damage scenario with depth of the stiffness losses ranges from 10% to 20%

**Fig. 2.** Two span beam and its damage patterns.

Table 1 lists the six natural frequencies of the damaged and intact conditions estimated from the proposed method and the technique of Kokot and Zembaty [27]. The information in this table are suitable for the comparison of these methods.

<table>
<thead>
<tr>
<th>Description of the damaged pattern</th>
<th>( f_1 ) (Hz)</th>
<th>( f_2 ) (Hz)</th>
<th>( f_3 ) (Hz)</th>
<th>( f_4 ) (Hz)</th>
<th>( f_5 ) (Hz)</th>
<th>( f_6 ) (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intact [27]</td>
<td>4.48</td>
<td>7.00</td>
<td>17.93</td>
<td>22.70</td>
<td>40.38</td>
<td>47.41</td>
</tr>
<tr>
<td>Intact (Present work)</td>
<td>4.48</td>
<td>7.00</td>
<td>17.94</td>
<td>22.75</td>
<td>40.39</td>
<td>47.50</td>
</tr>
<tr>
<td>5% &amp; 10% damage scenario [27]</td>
<td>4.31</td>
<td>6.71</td>
<td>17.39</td>
<td>21.90</td>
<td>37.19</td>
<td>43.62</td>
</tr>
<tr>
<td>5% &amp; 10% damage scenario (Present work)</td>
<td>4.31</td>
<td>6.71</td>
<td>17.40</td>
<td>21.94</td>
<td>38.84</td>
<td>45.66</td>
</tr>
<tr>
<td>10% &amp; 20% damage scenario [27]</td>
<td>4.12</td>
<td>6.40</td>
<td>16.82</td>
<td>21.05</td>
<td>38.83</td>
<td>45.57</td>
</tr>
<tr>
<td>10% &amp; 20% damage scenario (Present work)</td>
<td>4.13</td>
<td>6.40</td>
<td>16.84</td>
<td>21.10</td>
<td>37.22</td>
<td>43.74</td>
</tr>
</tbody>
</table>

It is obvious that the natural frequencies obtained from the method introduced in section 4 is compatible with those gained by [27]. Therefore, the authors' formulation and their computer program are verified. Obviously, the occurrence of damage reduces the natural frequencies of the beam. This is because of the fact that damage decreases the stiffness of the structure. Now, the damage identification scheme is applied for estimating the severity and location of damage. For this purpose, various number of modes are applied. The estimated damage indices corresponding to the aforesaid complex damage scenarios are listed in Tables 2 and 3.

It is clear that the proposed damage identification approach is able to successfully find the magnitude and location of damage in this beam by using only few numbers of mode shapes and natural frequencies.
Table 2. The calculated damage indices corresponding to 5% & 10% damage scenario.

<table>
<thead>
<tr>
<th>Element number</th>
<th>5% &amp; 10% damage scenario [27]</th>
<th>5% &amp; 10% damage scenario (Present work)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>By using the first 2 eigen-pairs</td>
</tr>
<tr>
<td>1</td>
<td>0.90</td>
<td>0.43</td>
</tr>
<tr>
<td>2</td>
<td>0.95</td>
<td>0.68</td>
</tr>
<tr>
<td>3</td>
<td>0.95</td>
<td>1.04</td>
</tr>
<tr>
<td>4</td>
<td>0.90</td>
<td>1.32</td>
</tr>
<tr>
<td>5</td>
<td>0.90</td>
<td>1.18</td>
</tr>
<tr>
<td>6</td>
<td>0.95</td>
<td>0.70</td>
</tr>
<tr>
<td>7</td>
<td>0.95</td>
<td>0.38</td>
</tr>
<tr>
<td>8</td>
<td>0.90</td>
<td>0.53</td>
</tr>
<tr>
<td>9</td>
<td>0.90</td>
<td>0.78</td>
</tr>
<tr>
<td>10</td>
<td>0.95</td>
<td>0.73</td>
</tr>
<tr>
<td>11</td>
<td>0.95</td>
<td>0.66</td>
</tr>
<tr>
<td>12</td>
<td>0.90</td>
<td>0.96</td>
</tr>
<tr>
<td>13</td>
<td>0.90</td>
<td>1.30</td>
</tr>
<tr>
<td>14</td>
<td>0.95</td>
<td>1.09</td>
</tr>
<tr>
<td>15</td>
<td>0.95</td>
<td>0.46</td>
</tr>
<tr>
<td>16</td>
<td>0.90</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 3. The calculated damage indices corresponding to 10% & 20% damage scenario.

<table>
<thead>
<tr>
<th>Element number</th>
<th>10% &amp; 20% damage scenario [27]</th>
<th>10% &amp; 20% damage scenario (Present work)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>By using the first 2 eigen-pairs</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.39</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>0.62</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>0.95</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>1.21</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>1.09</td>
</tr>
<tr>
<td>6</td>
<td>0.9</td>
<td>0.65</td>
</tr>
<tr>
<td>7</td>
<td>0.9</td>
<td>0.35</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>0.48</td>
</tr>
<tr>
<td>9</td>
<td>0.8</td>
<td>0.70</td>
</tr>
<tr>
<td>10</td>
<td>0.9</td>
<td>0.67</td>
</tr>
<tr>
<td>11</td>
<td>0.9</td>
<td>0.62</td>
</tr>
<tr>
<td>12</td>
<td>0.8</td>
<td>0.89</td>
</tr>
<tr>
<td>13</td>
<td>0.8</td>
<td>1.19</td>
</tr>
<tr>
<td>14</td>
<td>0.9</td>
<td>0.99</td>
</tr>
<tr>
<td>15</td>
<td>0.9</td>
<td>0.41</td>
</tr>
<tr>
<td>16</td>
<td>0.8</td>
<td>0.07</td>
</tr>
</tbody>
</table>

6.2. A Simply-Supported Beam

Here, a simply supported beam mentioned in Gharighoran et al. [21] is assessed, which includes 10 beam elements. The bending stiffness, cross sectional area, mass density and span length of this beam are assumed to be equal to those of the previous example.

To model the simple supports, translational springs with the stiffness of $10^{+12}$ kg/m are
used. To consider damping for this beam, it is assumed that $\alpha = 4.48$ and $\beta = 4.3 \times 10^{-4}$.

A complex damage scenario is considered. This beam and its damage pattern are illustrated in Figure 3. Damage detection is conducted with and without noise.

Now, the first five natural frequencies of the damaged and sound beam are computed. The obtained results are compared with those of [21] in Table 4 for the verification.

![Fig. 3. The simply supported beam with its damage pattern.](image)

Table 4. The natural frequencies of the undamaged and damaged simply supported beam.

<table>
<thead>
<tr>
<th>Description of the damaged pattern</th>
<th>$f_1$ (Hz)</th>
<th>$f_2$ (Hz)</th>
<th>$f_3$ (Hz)</th>
<th>$f_4$ (Hz)</th>
<th>$f_5$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intact (Present work )</td>
<td>1.58</td>
<td>6.31</td>
<td>14.24</td>
<td>25.31</td>
<td>39.88</td>
</tr>
<tr>
<td>Intact (Present work + Damping)</td>
<td>1.58+0.048i</td>
<td>6.31+0.041i</td>
<td>14.24+0.076i</td>
<td>25.31+0.13i</td>
<td>39.89+0.2i</td>
</tr>
<tr>
<td>Damaged (Present work + Damping)</td>
<td>1.26+0.056i</td>
<td>5.32+0.038i</td>
<td>12.48+0.067i</td>
<td>21.87+0.11i</td>
<td>34.33+0.17i</td>
</tr>
</tbody>
</table>

It is clear that the proposed approach is successful in finding the natural frequencies. Obviously, damage leads to a reduction in stiffness. As a result, the natural frequencies of the damaged beam are less than the second one. At this stage, the first five mode shapes of the damaged and intact beam are illustrated in Figure 4.

![Fig. 4. The damaged and undamaged mode shapes.](image)
Herein, the suggested damage detection method is applied to estimate the damage indices of this structure. For this purpose, only the first five natural frequencies and mode shapes of the damaged structure are employed. The obtained results are presented in Figure 5.

![Detected vs. exact damage indices](image)

**Fig. 5.** Detected vs. exact damage indices.

Accordingly, the proposed method is capable of estimating the severity and location of damage in this beam structure with and without noise.

6.3. A Fixed Support Beam

In this subsection, a fixed support beam with $0.3m \times 0.3m$ cross-section is evaluated. It is made of concrete with Young’s modulus 25 GPa and concrete density $\rho = 2500 \text{ kg/m}^3$. It should be added that the beam length is 10m. To model the fixed supports, translational and rotational springs are used. Their stiffness equal to $10^{+12} \text{kg/m}$ and $10^{+12} \text{kg.m/m}$, respectively. To consider damping for this beam, it is assumed that $\alpha = 4.48$ and $\beta = 4.3 \times 10^{-4}$.

It is assumed that this beam is divided into 10 elements, and five of them are damaged. Similarly, damage detection is conducted with and without noise.

This beam and its damage pattern are shown in Figure 5.

![The fixed support beam with its damage pattern](image)

**Fig. 6.** The fixed support beam with its damage pattern.

Now, its first five natural frequencies are listed in Table 6 for both the damaged and undamaged cases. For this purpose, the method presented in section 4 is deployed.
Table 6. The natural frequencies of the undamaged and damaged fixed support beam.

<table>
<thead>
<tr>
<th>Description of the damaged pattern</th>
<th>$f_1$ (Hz)</th>
<th>$f_2$ (Hz)</th>
<th>$f_3$ (Hz)</th>
<th>$f_4$ (Hz)</th>
<th>$f_5$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intact</td>
<td>9.84</td>
<td>27.37</td>
<td>53.95</td>
<td>89.99</td>
<td>135.74</td>
</tr>
<tr>
<td>Intact + Damping</td>
<td>9.83+0.055i</td>
<td>27.36+0.14i</td>
<td>53.95+0.27i</td>
<td>89.99+0.45i</td>
<td>135.74+0.68i</td>
</tr>
<tr>
<td>Damaged + Damping</td>
<td>9.49+0.054i</td>
<td>26.43+0.13i</td>
<td>52.48+0.26i</td>
<td>85.97+0.43i</td>
<td>128.86+0.64i</td>
</tr>
</tbody>
</table>

Mode Shapes Damage(DS) and Sound Stage(SS)

Fig. 7. The damaged and undamaged mode shapes.

Moreover, Figure 7 shows the first five mode shapes of this beam after and prior to damage.

Now, with the help of these five natural frequencies and mode shapes, the proposed damage detection technique calculates the damage indices. The obtained results are tabulated in Figure 8.

Obviously, the authors' technique is able to find the magnitude and location of damage in this beam with high accuracy.

Fig. 8. Detected vs. exact damage indices.
7. Conclusions

Most analytical schemes of damage identification used finite element formulation and the only limited number of them were developed based on Rayleigh–Ritz approach. This paper proposed a new damage detection method based on Rayleigh–Ritz technique. The proposed method is applicable to beam-like structures with any kind of boundary condition, and it only requires a few number of natural frequencies and modes shapes of the damaged structure. In this process, lumped rotational and translational springs were utilized for modeling various boundary conditions. Moreover, the proposed method is capable of considering damping. The high robustness and efficiency of the suggested approach was proved by performing different numerical examples. Also, in these examples, the effect of noisy measurement on responses was assessed. The results showed that the proposed approach was successful in damage detection of bridges when the noisy data are available.

REFERENCES


[23] Doebbling, S.W., et al., Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: a literature review, in Research report LA-13070-MS, ESA-EA Los Alamos National Laboratory. 1996: Los Alamos, NM, USA.


