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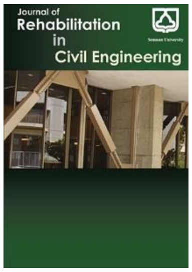


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# New Methods for Dynamic Analysis of Structural Systems under Earthquake Loads

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**ABSTRACT**

Two numerical methods are proposed for dynamic analysis of single-degree-of-freedom systems. Basics of dynamics and elementary tools from numerical calculus are employed to formulate the methods. The energy conservation principles triggered the basic idea of the first method, so-called energy-based method (EBM). It is devised for dynamic analysis of linear damped system whose damping ratio is greater than 1%. The second method uses function approximation theory and integration scheme, and called simplified integration method (SIM). Several numerical examples are investigated through SIM. A detailed comparison is made between the proposed methods and the conventional ones. The results show that the proposed methods can estimate the dynamic response of linear damped systems with high accuracy. In the first example, the peak displacement is obtained 6.8747 cm and 6.8290 cm which closely approximate the highly exact response of Duhamel integral. Results show that Newmark-β method is the fastest one whose run-time is 0.0019 sec. EBM and SIM computational times are 0.0722 sec and 0.0021sec, respectively. SIM gives more accurate estimate and convergence rate than Newmark-β method. The difference of peak displacement obtained from two methods is almost less than 1%. Thus, SIM reliably estimates the dynamic response of systems with less computational cost.

## 1. Introduction

In structural dynamics, a multi-degree-of-freedom (MDOF) structure is often equalized by a single-degree-of-freedom (SDOF) model [31]. Such a SDOF system is based on the dynamic properties adopted from the MDOF system. Structural responses of this SDOF model to earthquake is determined by conducting a nonlinear time-history analysis of the model, subjected to a set of ground motion records. Accordingly, the numerical

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analysis of SDOF systems is of high importance in this field of structural engineering.

The dynamic response of SDOF system, subjected to an arbitrary time-dependent load function, is typically dominated by a second-order ordinary differential equation (ODE) [19]. It is technically known as governing differential equation of motion (GDEM).

The dynamic analysis of a SDOF system is possible by numerically solving the GDOM at successive instances. Such an analysis is often costly and time-consuming [7, 11]. Hence, it has been a place of challenge in the field of vibration engineering from very early. Several approximate numerical methods have been developed by researchers in the recent decades: Wilson-$\theta$ [32], Newmark-$\beta$ algorithm [24], HHT-$\alpha$ method [16], WBZ-$\alpha$ method [33], $\rho$-method [4], HP-$\theta 1$ method [18], Duhamel integral method [9], and piecewise exact [8] procedures. These methods are categorized according to their formulation platform.

In general, numerical methods are classified into two main categories. First, the methods in which the dynamic response of structure is determined by summing the dynamic response of finite short impulse loads. These methods are based on the superposition principle. They are limited to linear elastic systems and cannot be applied to inelastic systems [9, 10, 29, 30]. The commonly used Laplace and Fourier methods fall into this category [15, 27, 14]. Second, the stepwise numerical methods which involve nonlinearities of mass, stiffness, and damping. Numerous stepwise methods are available in the literature. Piecewise exact method, Runge-Kutta, Newmark-$\beta$, Houbolt, Wilson-$\theta$ [8, 3, 5], finite difference, Euler-Gauss methods belong to this group [1, 12, 13, 25, 26, 28]. These methods divide the time domain into a sequence of small intervals, so-called time steps, and determine the response of the system using integration method. These methods often work well with any type of nonlinearities.

Recently, a simplified method was proposed by Li and Wu [22] to determine the dynamic response of inelastic SDOF systems with the time-varying mass and stiffness parameters. Wu and Lim [34] proposed an iterative algorithm solution to estimate the nonlinear response of SDOF systems possessing general nonlinear restoring behavior. They also proposed a new analytical method for calculation of the natural period of SDOF systems. In another study, some serious attempts have been made by Chang [6] to demonstrate the accuracy and efficiency of the Newmark method. Kazakov [20] studied the response of SDOF systems using the Duhamel integral for some special loading cases. The results showed that the Duhamel integral method is accurate enough in estimating the dynamic response of linear damped and undamped SDOF systems. A simple numerical method was developed by Kurt and Çevik [21], in which the Taylor polynomial was employed to estimate the dynamic response of SDOF systems. The accuracy and efficiency of some stepwise numerical procedures such as Wilson-$\theta$, Newmark-$\beta$ algorithm, central difference method, Runge-Kutta method, and Duhamel methods have been evaluated by Mohammadzadeh et al. [23].

In this paper, two new methods are introduced to compute the response of the SDOF systems to earthquake loads. Different formulation base is implemented to achieve simple algorithms dynamic analysis. Against conventional method like Newmark-$\beta$, the same procedure is used for main computation and iteration. This provides extreme simplicity for computer programming of linear and nonlinear analysis of vibration. The proposed methods benefit the basics of dynamics and elementary tools from numerical calculus. The first method, so-called energy-based method (EBM), is extended for linear damped SDOF systems whose damping ratios are almost greater than 1%. The second method is a more general one which is called simplified integration method (SIM). It can robustly conduct nonlinear analysis of various systems. Both methods are employed to analyze several general systems. A detailed comparison of the obtained results shows that the new methods accurately estimate the dynamic response of linear and nonlinear systems.

## 2. Problem statement

Fig. 1 shows an idealized linear mass-spring system under an arbitrary external force $F(t)$. This system is composed of three main components: mass, spring, and damper. Each component may have linear or nonlinear behavior. Real systems often experience nonlinearity during vibration. This nonlinearity is often originated from the spring behavior at large displacements or the non-viscous damping property of the system. Hence, the force induced by each component will be identified by a piecewise function of displacement and/or velocity.

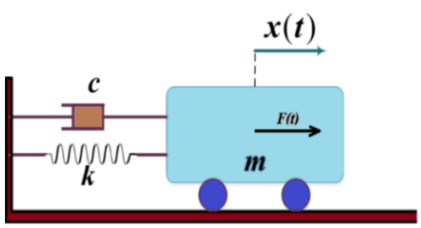


**Fig. 1.** Idealized mass-spring model of vibration under external force.

Denoting the relative displacement (or deformation) by $x$, velocity by $\dot{x}$, and acceleration by $\ddot{x}$, GDEM of a nonlinear system subjected to the external force $F(t)$ can be written as follows:

$$F_s(x,\dot{x}) + F_d(x\,,\dot{x}) + m(t)\,\ddot{x} = F(t) \quad (1)$$

where $F_s(x,\dot{x})$ is the spring force, $F_d(x,\dot{x})$ is the damping force. The mass of the system is $m(t)$. This is the most general form of vibration equation in which change of mass is included; although, it is not the case in structural dynamics. As shown in Fig. 2, when this system is subjected to the support acceleration of $a_g = a_g(t)$, the external force $F(t)$ should be replaced by $-ma_g(t)$.

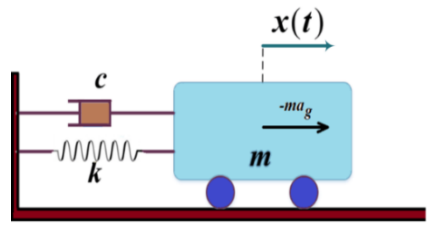


**Fig. 2.** Idealized mass-spring model of vibration under support excitation.

Hence, the GDEM of the nonlinear system under earthquake excitation is [26, 8]:

$$F_s(x,\dot{x}) + F_d(x\,,\dot{x}) + m(t)\,\ddot{x} = -m(t)\,a_g \quad (2)$$

Assuming linear behavior for spring, $F_s = kx$, and viscous damping, $F_D = c\dot{x}$, we can simplify Eq. (1) as:

$$kx + c\dot{x} + m\ddot{x} = F(t) \quad (3)$$

where $k$ is the constant of the weightless spring, $c$ is the viscous damping coefficient, $m$ is the constant mass of the system. For linear systems subjected to the earthquake loading, Eq. (2) can be presented in a simpler form as follows:

$$kx + c\dot{x} + m\ddot{x} = -ma_g \quad (4)$$

Dividing Eq. (4) by mass $m$, GDEM of linear system subjected to the ground acceleration can be expressed as:

$$\omega_n^2 x + 2\zeta\omega_n\dot{x} + \ddot{x} = -a_g \quad (5)$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency of the oscillator, and $\zeta = c/(2m\omega_n)$ is the damping ratio. It is the alternative expression of the vibration equation given by Eq. (4). This equation indicates that two systems with the same natural frequencies and damping ratios have an identical response $x = x(t)$, regardless of being massive and stiffer relative to each other.

In structural dynamics, support excitation is the mostly stricken case rather than external force. Support excitation is a discretely valued function which is often given in at time instance $t_i$ where $i = 1$ to $N$. $N$ is the number of time instance; and, $N - 1$ denotes the number of time steps. The length of time steps is usually assumed constant, i.e., $\Delta t = t_{i+1} - t_i$. Now, in stepwise algorithms, the GDEM should be satisfied at $t_i$:

$$F_{s,i} + F_{d,i} + m_i\, \ddot{x}_i = -m_i a_{g,i} \tag{6}$$

where $F_{s,i} = F_s(x_i, \dot{x}_i)$ is the spring force, $F_{d,i} = F_d(x_i\, , \dot{x}_i)$ is damping force, and $m_i = m(t_i)$ is the mass of the system, and $a_{g,i} = a_g(t_i)$ is the support acceleration, all at time $i^{th}$ instance. The objective of the numerical method is to determine the response at $t_{i+1}$, knowing the response at the previous time instance $t_i$. The response should satisfy GDEM at $t_{i+1}$:

$$F_{s,i+1} + F_{d,i+1} + m_{i+1}\, \ddot{x}_{i+1} = -m_{i+1} a_{g,i+1} \tag{7}$$

where $F_{s,i+1} = F_s(x_{i+1}, \dot{x}_{i+1})$, $F_{d,i+1} = F_d(x_{i+1}\, , \dot{x}_{i+1})$, $m_{i+1} = m(t_{i+1})$, and $a_{g,i+1} = a_g(t_{i+1})$ are the spring force, damping force, mass, and support acceleration at $t_{i+1}$, respectively.

## 2.1. Energy-based method (EBM)

Consider a linear SDOF system subjected to the support excitation. To obtain a recursive formulation, we first recall the definition of the work $dU$ done by a force $F$ during the straight pass $dx$:

$$dU = F\, dx \tag{8}$$

Multiplying Eq. (4) by $dx$, we obtain:

$$kx\, dx + c\dot{x}dx + m\ddot{x}dx = -ma_g dx \tag{9}$$

Each term of Eq. (9) can be interpreted as the work or energy corresponding to its component. $kxdx$ is the differential energy value of the spring force, $c\dot{x}dx$ is the differential energy dissipated by friction force, $m\ddot{x}dx$ is the differential kinetic energy of the system, and $-ma_g dx$ can be considered as the differential work done by the external excitation. Indeed, this equation indicates that the work done by the earthquake force (or the energy input of earthquake excitation) is distributed among three components of the system. It is reserved in spring or dissipated by friction or converted to the kinetic energy of the system.

Now, refining Eq. (9), we prepare it for numerical integration. Recalling that the velocity of the system and the ground acceleration are defined by $\dot{x} = dx/dt$ and $a_g = dv_g/dt$; and, substituting them into Eq. (9), we get:

$$kx\, dx + c\dot{x}dx + m\ddot{x}dx = -mdv_g\dot{x} \tag{10}$$

where $v_g$ is the velocity of support. Dividing the last equation by $\dot{x}$ and noting that $dx/\dot{x} = dt$, we can integrate Eq. (10) from $i$ to $i + 1$:

$$\int_{t_i}^{t_{i+1}} kxdt + \int_{x_i}^{x_{i+1}} cdx + \int_{t_i}^{t_{i+1}} m\ddot{x}dt = \int_{v_{g,i}}^{v_{g,i+1}} -mdv_g \tag{11}$$

which leads to:

$$\frac{k\, \Delta t}{2}(x_i + x_{i+1}) + c(x_{i+1} - x_i) + + \frac{m\, \Delta t}{2}(\ddot{x}_i + \ddot{x}_{i+1}) = -m\big(v_{g,i+1} - v_{g,i}\big) \tag{12}$$

where $x_i$, $\ddot{x}_i$, and $v_{g,i}$ are displacement, acceleration, and support velocity at time instance $i$, respectively. In a similar way, $x_{i+1}$, $\ddot{x}_{i+1}$, and $v_{g,i+1}$ are defined at time instance $i+1$. Solving Eq. (12) for $x_{i+1}$, we obtain:

$$x_{i+1} = -[m(v_{g,i+1} - v_{g,i}) + \left(\frac{k\,\Delta t}{2} - c\right)x_i + \frac{m\,\Delta t}{2}(\ddot{x}_i + \ddot{x}_{i+1})]/(\frac{k\,\Delta t}{2} + c) \quad (13)$$

which provides the required relation for computing $x_{i+1}$ in an iterative process. Similarly, solving Eq. (12) for $\dot{x}_{i+1}$, we have:

$$\dot{x}_{i+1} = -[m(v_{g,i+1} - v_{g,i}) + \left(\frac{k\,\Delta t}{2} - c\right)x_i + (\frac{k\,\Delta t}{2} + c)\,x_{i+1}]/m + \dot{x}_i \quad (14)$$

Eqs. (13) and (14) coupled with Eq. (4) constitute the processing core of EBM in linear analysis of SDOF systems. The step-by-step procedure of EBM is summarized in Table 1. Despite the extreme simplicity of EBM, it provides satisfactorily precise response for linear damped systems.

## 2.2. Simplified integration method (SIM)

Direct use of numerical integration scheme coupled with function approximation theory leads to straightforward and efficient formulation for solving nonlinear GDEM.

Basically, all of the integration-based methods essentially benefit from similar mathematical background. This study presents one of the simplest versions which runs fast and works precisely.

We first consider a SDOF system in which all of the nonlinearities are active. The 3$^{rd}$-order derivative can be determined by the following finite difference formula:

$$\dddot{x}_{i+1} = \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t} \quad (15)$$

Having the value of $\dddot{x}_i$ and $\dddot{x}_{i+1}$, the best picture of acceleration function could be a 3$^{rd}$-order polynomial between $t_i$ and $t_{i+1}$:

$$\ddot{x}(t) = A_0 + A_1(t - t_i) + A_2(t - t_i)^2 + A_3(t - t_i)^3 \quad (16)$$

**Table 1.** EBM algorithm for linear analysis of SDOF systems.

1. Initialize with: *

$$i = 1,\quad t_1 = 0\ ,\quad x_1 = x(0)\ ,\ \dot{x}_1 = \dot{x}(0)\ ,\ \ddot{x}_1 = -(a_{g,1} + \omega_n^2\,x_1 + 2\omega_n\zeta\dot{x}_1)$$

2. Predict the response at $i+1$ (or set them all to zero):

$$x_{i+1} = x_i + \frac{\Delta t}{2}\,\dot{x}_i\ \ ,\ \dot{x}_{i+1} = \dot{x}_i + \frac{\Delta t}{2}\,\ddot{x}_i$$

3. Correct the system state using the following

$$\ddot{x}_{i+1} = -(a_{g,i+1} + \omega_n^2\,x_{i+1} + 2\omega_n\zeta\,\dot{x}_{i+1})$$

$$x_{i+1} = -\left[m(v_{g.i+1} - v_{g.i}) + \left(\frac{k\,\Delta t}{2} - c\right)x_i + \frac{m\,\Delta t}{2}(\ddot{x}_i + \ddot{x}_{i+1})\right]/(\frac{k\,\Delta t}{2} + c)$$

$$\dot{x}_{i+1} = -\left[m(v_{g.i+1} - v_{g.i}) + \left(\frac{k\,\Delta t}{2} - c\right)x_i + (\frac{k\,\Delta t}{2} + c)\,x_{i+1}\right]/m + \dot{x}_i$$

4. Repeat steps 3 and 4 until none of the precision criteria is met.
5. Set $i = i + 1$ and repeat steps 2 to 5 for the next time instance.

*In the case of external loads instead of support excitation, use the following relations in steps 1 and 3, respectively:

$$\ddot{x}_1 = [F_1 - (kx_1 + c\dot{x}_1)]/m$$

$$\ddot{x}_{i+1} = [F_{i+1} - (kx_{i+1} + c\dot{x}_{i+1})]/m$$

**Important note**: EBM does not yield reliable results for undamped systems and the systems with $\zeta \leq 0.01$. In these cases, SIM is advised. However, in most of the real cases, we have $\zeta \geq 0.01$.

To determine the coefficients $A_i$, $i = 0{\sim}3$, we constitute the set of equations from the given conditions at $i$ and $i$+1. Solving this set of equations for $A_i$, and substituting them into Eq. (16), we use $d\dot{x} = \ddot{x}\, dt$ and integrate Eq. (16) from $i$ to $i + 1$, we get:

$$\dot{x}_{i+1} = \dot{x}_i + \frac{\ddot{x}_i + \ddot{x}_{i+1}}{2}\Delta t + \frac{\dddot{x}_i - \dddot{x}_{i+1}}{12}\Delta t^2 \qquad (17)$$

This formula uses the Corrected Trapezoidal Integration Rule (CTIR) for anti-differentiation.

Now, we focus on the displacement. We have the acceleration value and its derivative at $i$ and $i + 1$ in addition to the velocities $\dot{x}_\mathrm{i}$ and $\dot{x}_{\mathrm{i+1}}$. With these details in hand, we can introduce the following function for velocity response between the end points:

$$\dot{x}(t) = B_0 + B_1(t - t_i) + B_2(t - t_i)^2 + B_3(t - t_i)^3 + B_4\,(t - t_i)^4 + B_5\,(t - t_i)^5 \qquad (18)$$

Eq. (18) is a 5th-order polynomial with six unknown coefficients. The coefficients can be obtained in a similar manner which is already discussed for the velocity component. Substituting these coefficients into Eq. (18) and integrating it from $\boldsymbol{i}$ to $\boldsymbol{i + 1}$, we get:

$$\boldsymbol{x_{i+1} = x_i + \frac{\dot{x}_i + \dot{x}_{i+1}}{2}\Delta t + \frac{\ddot{x}_i - \ddot{x}_{i+1}}{10}\Delta t^2 + \frac{\dddot{x}_i + \dddot{x}_{i+1}}{120}\Delta t^3} \qquad (19)$$

The dynamic response of a nonlinear system can then be obtained by iterative use of Eqs. (17), (19), and the following equation, which is obtained from Eq. (3):

$$\boldsymbol{\ddot{x}_{i+1} = -(m a_{g,i+1} + F_{s,i+1} + F_{d,i+1})/m_{i+1}} \qquad (20)$$

or from Eq. (1), as follows:

$$\boldsymbol{\ddot{x}_{i+1} = [F_{i+1} - (F_{s,i+1} + F_{d,i+1})]/m_{i+1}} \qquad (21)$$

These relations suffice to conduct nonlinear analysis; however, the convergence rate of SIM can be improved by using an appropriate initial prediction for displacement and velocity at $i + 1$. ***Tylor*** polynomial offers great predictions at this instance as follows:

$$x_{i+1} = x_i + \dot{x}_i\Delta t + \frac{\ddot{x}_i}{2}\Delta t^2 + \frac{\dddot{x}_i}{6}\Delta t^3 \qquad (22)$$

$$\boldsymbol{\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i\Delta t + \frac{\dddot{x}_i}{2}\Delta t^2} \qquad (23)$$

After obtaining the displacement $x_{i+1}$ and velocity $\dot{x}_{\mathrm{i+1}}$, the resisting force $F_{\mathrm{s,i+1}}$ and damping force $F_{d,i+1}$ can be estimated using the system current state. Table 2 presents the stepwise procedure for the nonlinear analysis of SDOF systems using SIM.

It is noted that the mass nonlinearity is also included in the formulation, even though it may not be the case in structural dynamics. It is usually assumed unchanged during vibration.

More investigation shows that SIM is able to track the response pass when the system components suddenly change their behavior. Thus, if the friction or spring components were suddenly omitted from the system, the algorithm immediately detects the change. Despite the EBM, SIM has no limitation on system properties.

Replacing $F_{s,i}$, $F_{s,i+1}$, $F_{d,i}$, and $F_{d,i+1}$ by $kx_i$, $kx_{i+1}$, $c\dot{x}_i$, and $c\dot{x}_{i+1}$, respectively, we obtain the linear version of SIM. The procedure of linear SIM is presented in Table 3 for SDOF systems.

### 2.3. Precision criteria

Iteration should be terminated by pre-specified conditions to avoid endless trivial cycle. One or more of these conditions should be met to stop the iteration. Some of the common criteria are as follows:

1. The unbalanced or residual force from Eq. (3) diminishes while iteration is in progress. The absolute value of this criterion can make our judgment base of precision. For nonlinear systems, we can write:

$$R_i = \left|F_{s,i+1} + F_{d,i+1} + m_i\ddot{x}_i + m_i a_{g,i}\right| \qquad (24)$$

For linear systems, it is:

$$R_i = \left|kx_i + c\dot{x}_i + m_i\ddot{x}_i + m_i a_{g,i}\right| \qquad (25)$$

Denoting the residual force of $j^{th}$ iteration at $i^{th}$ time instant by $R_i^{(j)}$, this criterion is controlled by:

$$\left|R_i^{(j)}\right| \le \varepsilon_R \qquad (26)$$

The tolerance value $\varepsilon_R$ is usually selected from $10^{-8}$ to $10^{-3}$.

2. If displacement changes $\Delta x_i^{(j)} = x_i^{(j)} - x_i^{(j-1)}$, corresponding to $j^{th}$ iteration of $i^{th}$ time instant, falls in range, i.e.:

$$\left|\Delta x_i^{(j)}\right| \le \varepsilon_d \qquad (27)$$

where tolerance $\varepsilon_d$ is a positive value ranging from $10^{-8}$ to $10^{-3}$.

**Table 2.** SIM algorithm for nonlinear analysis of SDOF systems.

1. Initialize with: *

$$i = 1\ ,\ t_1 = 0\ ,\ x_1 = x(0)\ ,\ \dot{x}_1 = \dot{x}(0)$$
$$F_{s,1} = F_s(x_1, \dot{x}_1)\ ,\ F_{d,1} = F_d(x_1, \dot{x}_1)$$
$$\ddot{x}_1 = -(F_{s,1} + F_{d,1} + m_1 a_{g,1})/m_1$$
$$\dddot{x}_1 = 0$$

2. Predict the response at $i+1$ (or set them all to zero):

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i\,\Delta t + \frac{\dddot{x}_i}{2}\,\Delta t^2$$
$$x_{i+1} = x_i + \dot{x}_i\,\Delta t + \frac{\ddot{x}_i}{2}\Delta t^2 + \frac{\dddot{x}_i}{6}\,\Delta t^3$$

3. Iterate the following until none of the convergence criteria is met:

$$\ddot{x}_{i+1} = -\left(m a_{g,i+1} + F_{s,i+1} + F_{d,i+1}\right)/m_{i+1}$$
$$\dddot{x}_{i+1} = \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t}$$
$$\dot{x}_{i+1} = \dot{x}_i + \frac{\ddot{x}_i + \ddot{x}_{i+1}}{2}\Delta t + \frac{\dddot{x}_i - \dddot{x}_{i+1}}{12}\Delta t^2$$
$$x_{i+1} = x_i + \frac{\dot{x}_i + \dot{x}_{i+1}}{2}\,\Delta t + \frac{\ddot{x}_i - \ddot{x}_{i+1}}{10}\Delta t^2 + \frac{\dddot{x}_i + \dddot{x}_{i+1}}{120}\Delta t^3$$
$$F_{s,i+1} = F_s(x_{i+1}, \dot{x}_{i+1})\ ,\ F_{d,i+1} = F_d(x_{i+1}, \dot{x}_{i+1})$$

4. Set $i = i + 1$ and repeat steps 2 to 4 for the next time instance.

*In the case of external loads instead of support excitation, use the following relations in steps 1 and 3, respectively:

$$\ddot{x}_1 = [F_1 - \left(F_{s,1} + F_{d,1}\right)]/m_1$$
$$\ddot{x}_{i+1} = [F_{i+1} - (F_{s,i+1} + F_{d,i+1})]/m_{i+1}$$

3. Incremental work done by the residual force $R_i^{(j)} = R_i^{(j)} - R_i^{(j-1)}$ through the displacement change $\Delta x_i^{(j)} = x_i^{(j)} - x_i^{(j-1)}$ becomes less than its tolerance value $\varepsilon_w$ [8]:

$$\frac{1}{2}\left|\Delta x_i^{(j)} R_i^{(j)}\right| \leq \varepsilon_w \tag{28}$$

The value of $\varepsilon_w$ is chosen near to the computer precision, the smallest positive value recognizable, since the left side of this inequality is the product of two infinitesimal quantities.

## 2.4. Approaches to increase precision

There are various approaches to increase the accuracy of response with stepwise algorithms. Some of them are as follows:

- For the excitation given in the form of a continuous analytical function, use of fine mesh for time steps is an efficient way to increase precision of response.

**Table 3.** SIM algorithm for linear analysis of SDOF systems.

1. Initialize with *

$$\mathrm{i} = 1 \ , \ t_1 = 0 \ , \ x_1 = x(0) \ , \ \dot{x}_1 = \dot{x}(0)$$
$$\ddot{x}_1 = -(m a_{g,1} + k x_1 + c\dot{x}_1)/m$$
$$\dddot{x}_1 = 0$$

2. Predict the response at $i + 1$ (or set them all to zero):

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i\, \Delta t + \frac{\dddot{x}_i\, \Delta t^2}{2}$$
$$x_{i+1} = x_i + \dot{x}_i\, \Delta t + \frac{\ddot{x}_i \Delta t^2}{2} + \frac{\dddot{x}_i\, \Delta t^3}{6}$$

3. Iterate the following until none of the convergence criteria is met:

$$\ddot{x}_{i+1} = -(m a_{g,i+1} + k x_{i+1} + c\dot{x}_{i+1})/m$$
$$\dddot{x}_{i+1} = \frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t}$$
$$\dot{x}_{i+1} = \dot{x}_i + \frac{\ddot{x}_i + \ddot{x}_{i+1}}{2}\Delta t + \frac{\dddot{x}_i - \dddot{x}_{i+1}}{12}\Delta t^2$$
$$x_{i+1} = x_i + \frac{\dot{x}_i + \dot{x}_{i+1}}{2}\,\Delta t + \frac{\ddot{x}_i - \ddot{x}_{i+1}}{10}\Delta t^2 + \frac{\dddot{x}_i + \dddot{x}_{i+1}}{120}\Delta t^3$$

4. Set $i = i + 1$ and repeat steps 2 to 4 for the next time instance.

*In the case of external loads instead of support excitation, use the following relations in steps 1 and 3, respectively:

$$\ddot{x}_1 = [F_1 - (k x_1 + c\dot{x}_1)]/m$$
$$\ddot{x}_1 = [F_{i+1} - (k x_{i+1} + c\dot{x}_{i+1})]/m$$

- If the values of $F(t)\ or - m\ddot{a}_g$ are only given (or sampled in advanced) at discrete time instances, we can minify the length of sampling time steps, linearly interpolating its values for a finer mesh. Linear interpolation does not add redundant information to the excitation data; nevertheless, the algorithm works more exactly with finer mesh.
- In the case of earthquake excitation, spline interpolation of earthquake record for finer mesh almost yields reliable results; although,

this may slightly manipulate the frequency content of the excitation function.

## 3. Examples

Here, several numerical examples are investigated to show the generality and efficiency of the presented algorithms. The results are compared with some of the conventional methods. It is noted that the systems are loaded by earthquake excitation rather than other simple functions. It is evident that if the proposed techniques work appropriately with irregular earthquake excitation, they can certainly handle simpler cases as well as step, ramp, or even impulse load functions.

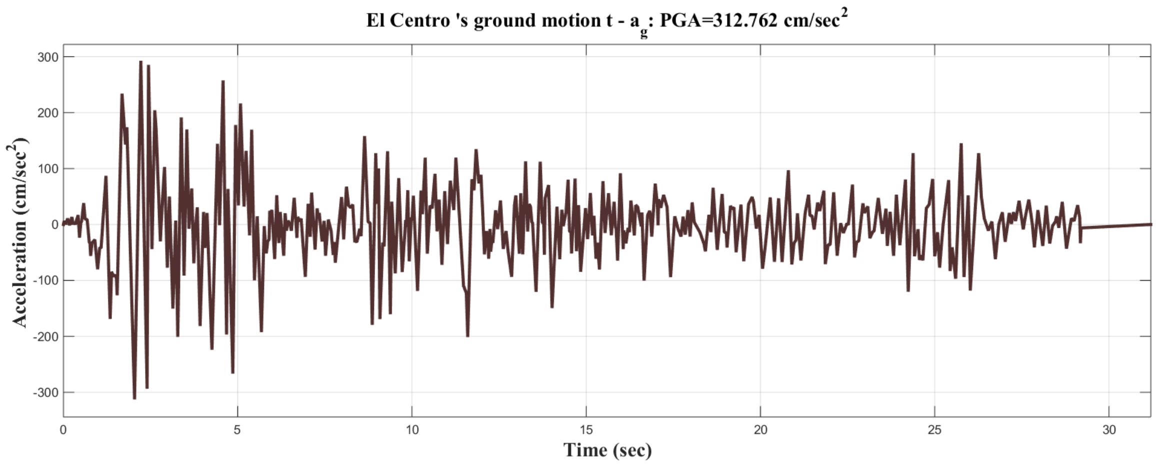


**Fig. 3.** El Centro's ground motion record.

### 3.1. Example I: Linear damped system under El centro earthquake

The performance of the EBM and SIM is studied on a linear system subjected to El Centro ground motion record (Fig. 3). The peak ground acceleration (PGA) is 0.32g.

The system is initially at rest. The properties of the system are: mass $m = 45.594\ kg = 0.45594\ kN.sec^2/cm$, natural period $T_n = 0.5\ sec$, damping ratio $\zeta = 0.02$, and stiffness $k = 72\ kN/cm$. Time step of the earthquake record is $\Delta t = 0.02\ sec$. The dynamic response of this system is obtained by EBM and SIM, Duhamel integral, and Newmark-β methods. The responses are plotted in Fig. 4. All methods yield approximately close responses. New methods properly detect the response curve with a time increment of $\Delta t = 0.02\ sec$. As expected, Duhamel integral works more precisely; however, it is very time consuming.

To show the convergence of new methods, we re-analyze the same problem using a finer mesh of $\Delta t = 0.002$. The obtained results are graphically presented in Fig. 5. It is noted that Duhamel, Newmark-$\beta$, and simplified integration methods all provide almost identical responses with insignificant differences.

The peak response values and run-time of the methods are summarized in Table 4. According to Table 4, Duhamel integral is the slowest one and

Newmark-$\beta$ method is the fastest method. However, SIM also runs fast compared to Newmark-$\beta$.

**Table 4.** Peak response values and run times of linear analyses in example I with $\Delta t = 0.002$ sec.

| item \ method | Duhamel integral method | Newmark-β method | Presented EBM | Presented SIM |
|---|---|---|---|---|
| Max Disp. (cm) | 6.8277 | 6.8272 | 6.8747 | 6.8290 |
| Max Vel. (cm/sec) | NC* | 81.954 | 82.418 | 81.908 |
| Max Accel. (cm/sec$^2$) | NC* | 1235.51 | 1240.84 | 1234.80 |
| Number of iteration | 1 | 1 | 5 | 2 |
| Run time (sec) | 13.005 | 0.0019 | 0.0722 | 0.0021 |

*NC: Not compute

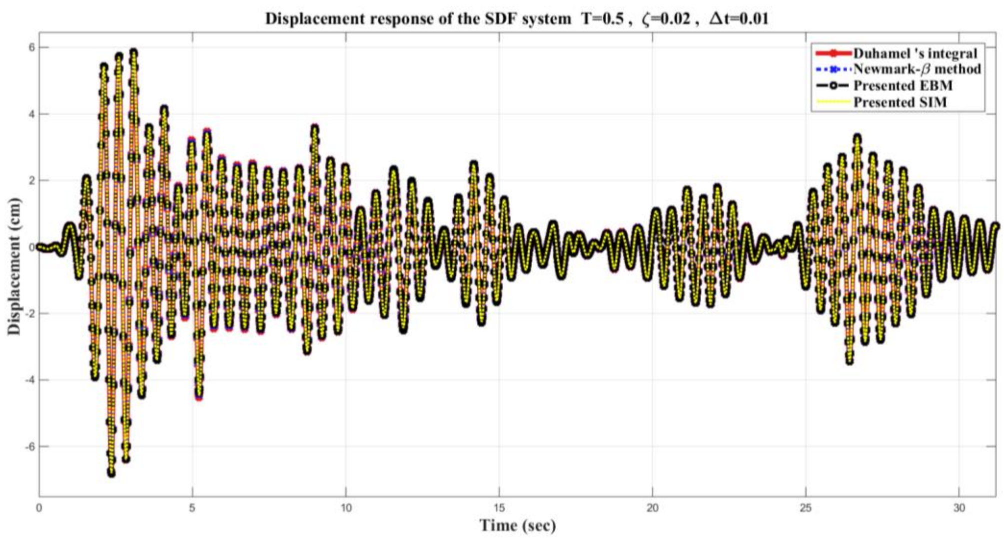


**Fig. 4.** Displacement responses for the linear SDOF system in example I (Coarse mesh).

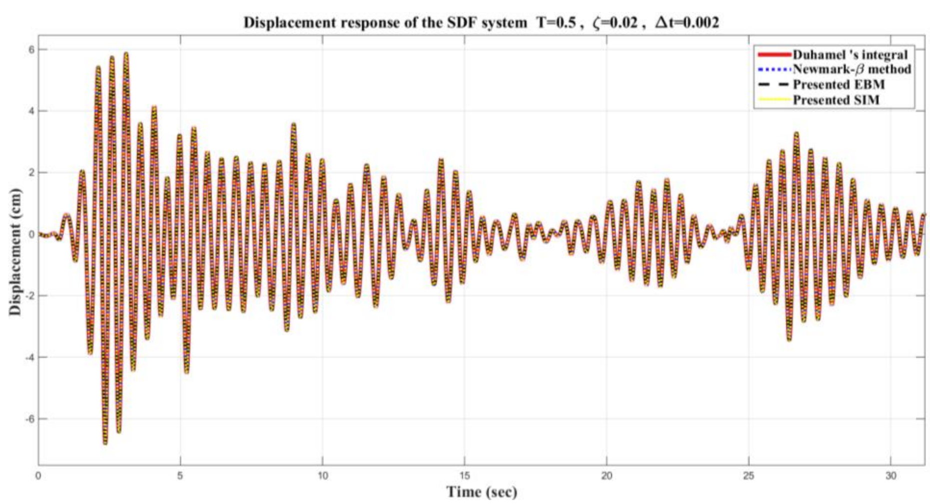


**Fig. 5.** Displacement responses for the linear SDOF system in example I (Fine mesh).

### 3.2. Example II: Linear damped system under Kobe earthquake

This example investigates another linear undamped system to study stability and convergence of new methods. Kobe ground motion record (Fig. 6) with peak ground acceleration $PGA = 0.34g$ is applied to the system which is initially at rest. The properties of the system are: mass $m = 45.594\ kg = 0.45594\ kN.sec^2/cm$, natural period $T_n = 0.3\ sec$, damping ratio $\zeta = 0.05$, and stiffness $k = 200\ kN/cm$. Time step of the earthquake record is $\Delta t = 0.02\ sec$. The system is analyzed by various methods and the responses are plotted and compared in Fig. 7. Duhamel integral is identified as the most exact response. New methods have very small deviation from Duhamel integral.

Their stabilities are evident in Fig. 7 and there is no distinct artificial damping with the presented responses.

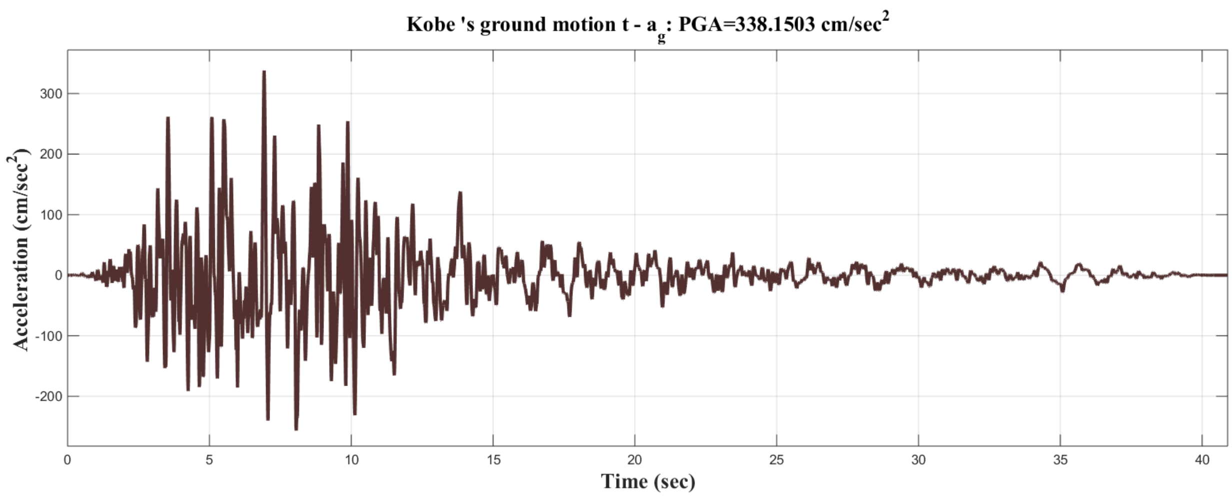


**Fig. 6.** Kobe’s ground motion record.

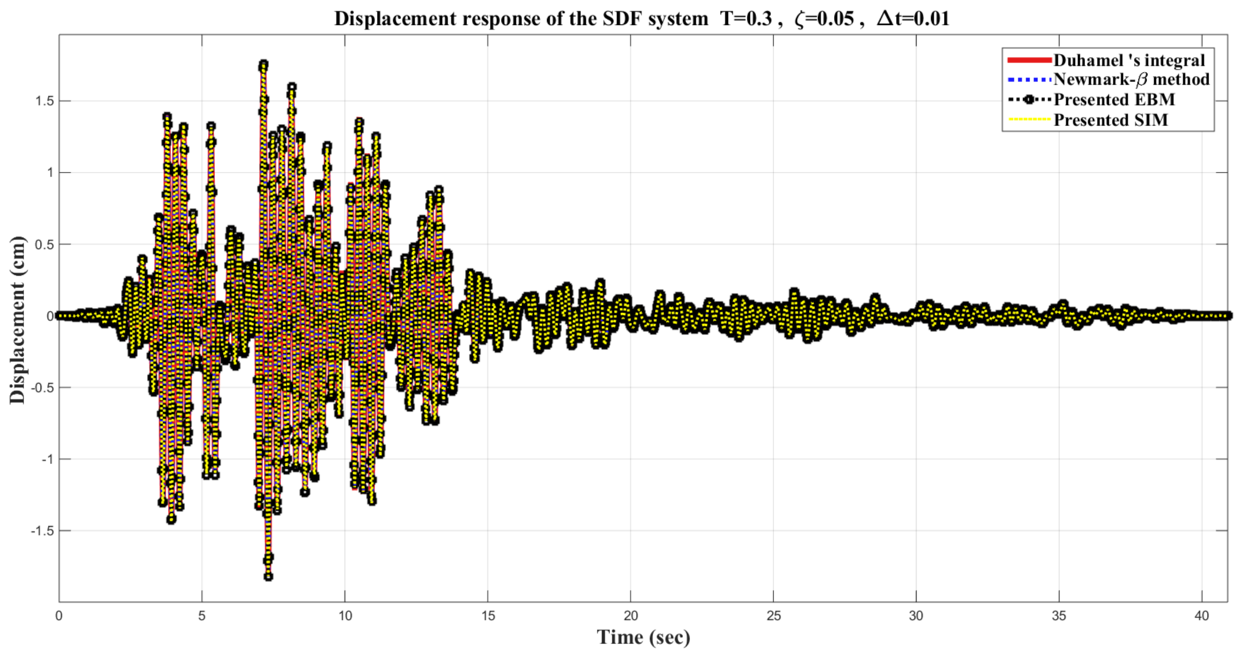


**Fig. 7.** Displacement responses for the linear SDOF system in example II.

**Table 5.** Peak response values and run times of linear analyses in example II with $\Delta t = 0.01$ sec.

| method / item | Duhamel integral method | Newmark-β method | Presented EBM | Presented SIM |
|---|---|---|---|---|
| Max Disp. (cm) | 1.8123 | 1.8182 | 1.8197 | 1.8171 |
| Max Vel. (cm/sec) | NC* | 34.613 | 34.553 | 34.593 |
| Max Accel. (cm/sec$^2$) | NC* | 720.86 | 717.09 | 720.45 |
| Number of iteration | 1 | 1 | 5 | 2 |
| Run time (sec) | 0.945 | 0.005 | 0.0722 | 0.005 |

*NC: Not computed

The peaks and the run time of each method are reported in Table 5. Examples I and II show that the proposed methods work properly with damped and undamped systems under earthquake excitation. Away from computational time, the capability of achieving satisfactory level of precision is admirable with the current algorithms.

## 3.2. Example III: Nonlinear undamped system under El Centro ground motion

This example conducts a nonlinear analysis using SIM. We should first compute the elastic response of the corresponding linear system to obtain the required information for nonlinear analysis. The system is subjected to El Centro ground motion record given by Fig. 3. The system properties are: mass $m = 45.594\,kg$, natural period $T_n = 0.5\,sec$, damping ratio $\zeta = 0$, stiffness $k = 72\,kN/cm$, and $\Delta t = 0.02\,sec$ time step. EBM cannot analyze this linear system because it is an undamped one. Hence, general method of SIM is implemented in this section. First, we analyze the system using Newmark-$\beta$ method to regenerate the solution presented in reference [8]. This response is shown in Fig. 8. Newmark-β parameters are set $\gamma = 1/2$ and β=1/6 to work with $\Delta t = 0.01 sec$ time step. The obtained response curve is plotted in Fig. 9. The peak displacement is computed 3.34 inch at $t = 26.36\, sec$. This is almost the same as that one reported in [8]. In Fig. 9, the overall layout of the presented response closely matches that one given in [8]. However, use of finer mesh size shows this response is not accurate at all. It means that Newmark-$\beta$ method needs finer steps to yield an accurate enough response in this case. To have a precise comparison, this system is re-analyzed with $\Delta t = 0.001 sec$ finer step size. The response is plotted in Fig. 10. A high level of coincidence is evident in this figure. Both Newmark-β method and SIM identify the peak at $t = 11.53\, sec$ instead of $t = 26.36\, sec$. Comparison of response in Fig. 9 and Fig. 10 shows that the peak displacement is $x_{max} = 8.200\, cm$, which occurs at $t = 11.53\, sec$, whereas it is reported $x_{max} = 8.483\, cm$ at $t = 26.35\, sec$ in [8]. Displacement, velocity, and acceleration responses are plotted in Fig. 11. Peak values are also given on the same plot. According to Fig. 11, the accurate maximum resisting force is:

$$f_o = k\, x_{max,elastic} = 70 \times 8.200 = 574\, kN \quad (29)$$

Nevertheless, to obtain comparable results with reference [8], we use the same value of $f_o$ which is given in this reference:

$$f_o = k\, x_{max.elastic} = 70 \times 8.48 = 593.6\, kN \quad (30)$$

It is slightly different from its exact value.

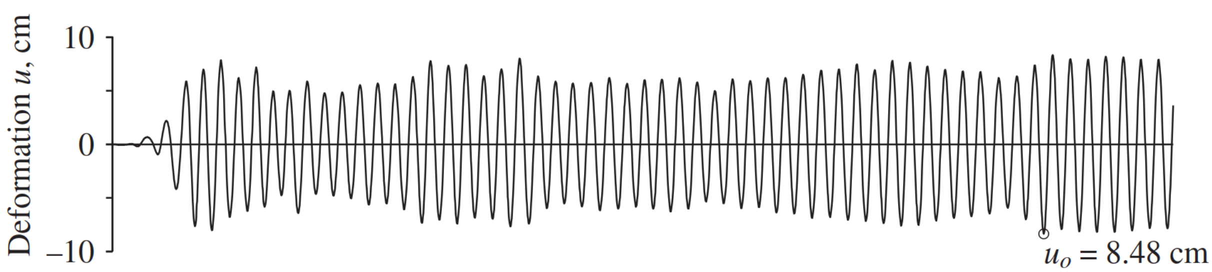


**Fig. 8.** Displacement response reported in [8] for the linear SDOF system in example III (Coarse mesh).

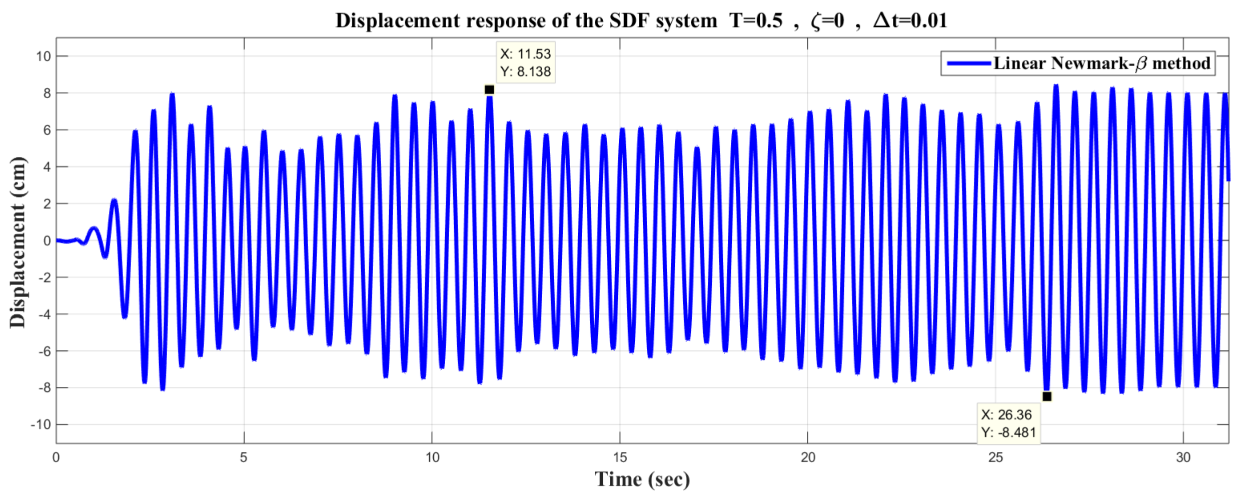


**Fig. 9.** Displacement response obtained from Newmark-β method for the linear SDOF system in example III (Coarse mesh).

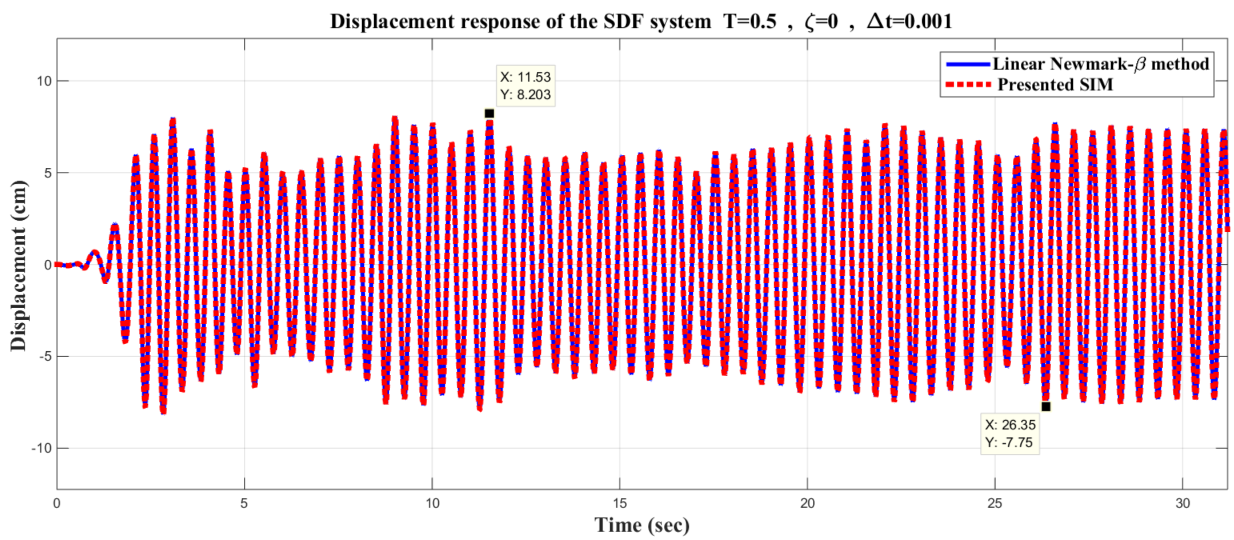


**Fig. 10.** Displacement response for the linear SDOF system in example III (Fine mesh).

Here, we consider the same system except that its resisting force follows the nonlinear behavior given by Fig 12. The magnitude of yielding force is assumed to be $0.125f_o$:

$$f_y = 0.125f_o = 0.125 \times 593.6 = 74.6\,kN \quad (31)$$

Fig. 13 is the displacement response from $t = 0{\sim}10\,sec$ which is reported in [8]; and, Fig. 14 shows the response obtained from SIM. An excellent agreement is evident between the presented approach and Newmark-β method. The peak values are also the same. All kinematic responses of the nonlinear system are plotted in Fig. 15. A summary report is also available in Table 6.

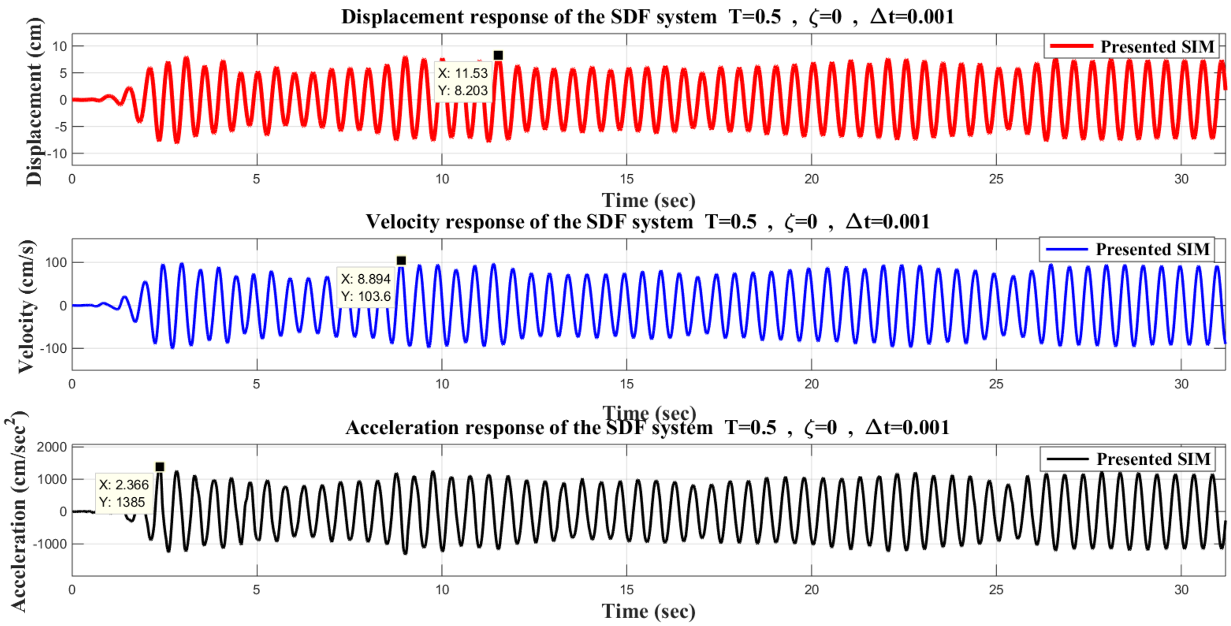


**Fig. 11.** Responses for the linear SDOF system in example III (Fine mesh).

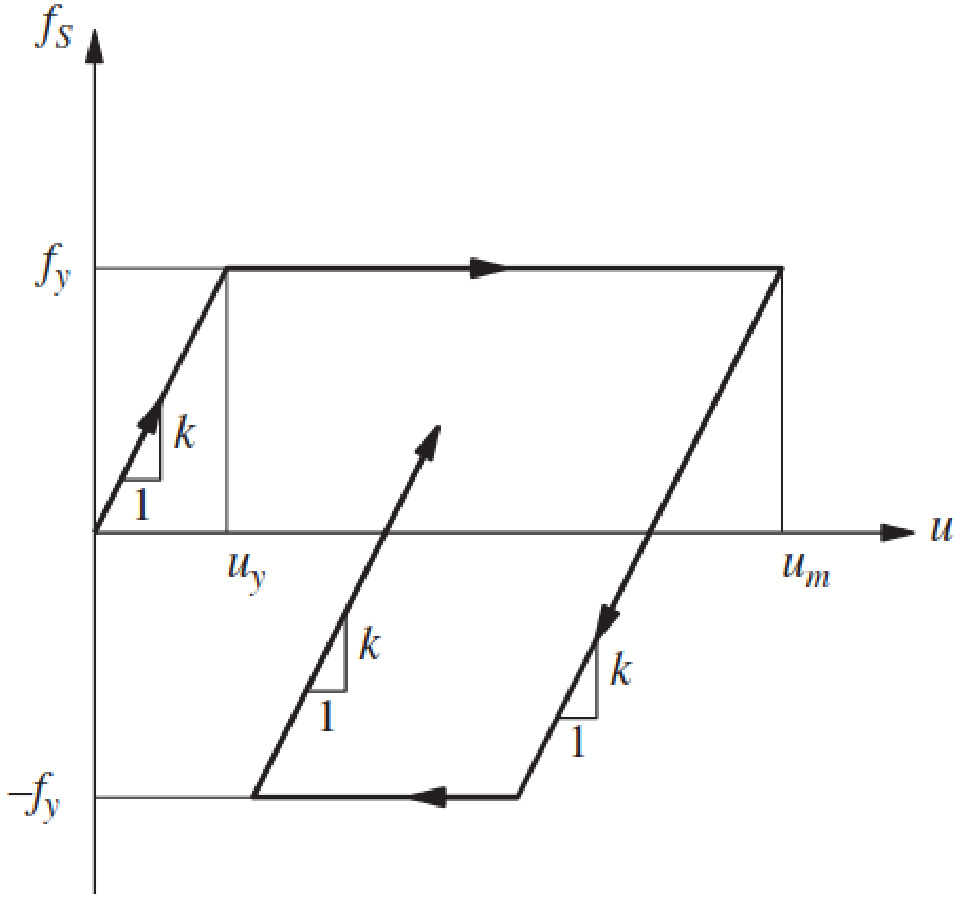


**Fig. 12.** Idealized elastoplastic behavior of restoring force-deformation component in example III.

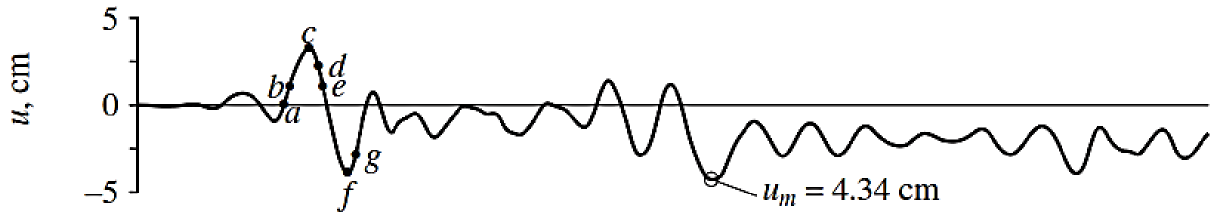


**Fig. 13.** Displacement response reported in [8] for the nonlinear SDOF system in example III.

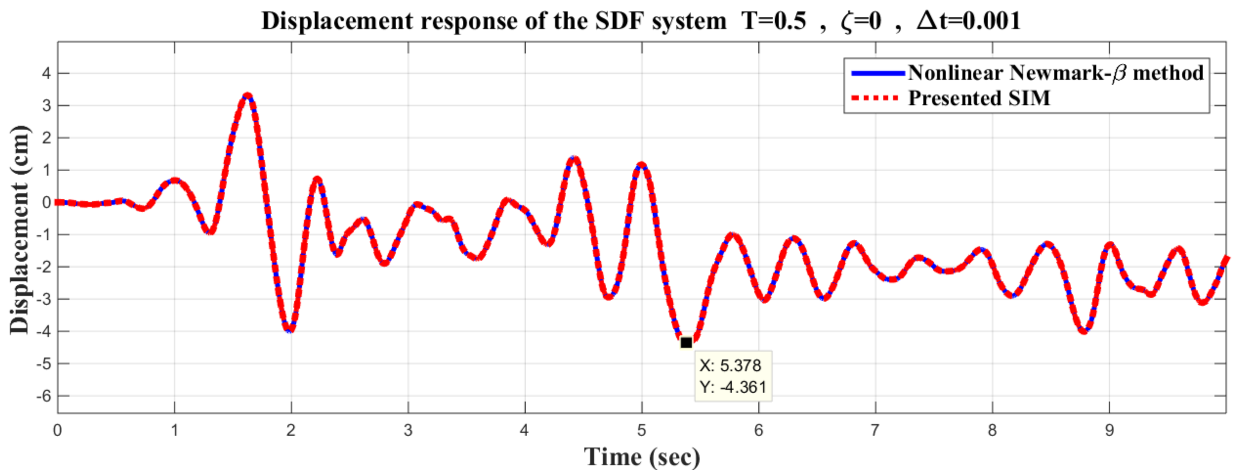


**Fig. 14.** Displacement response for the nonlinear SDOF system in example III.

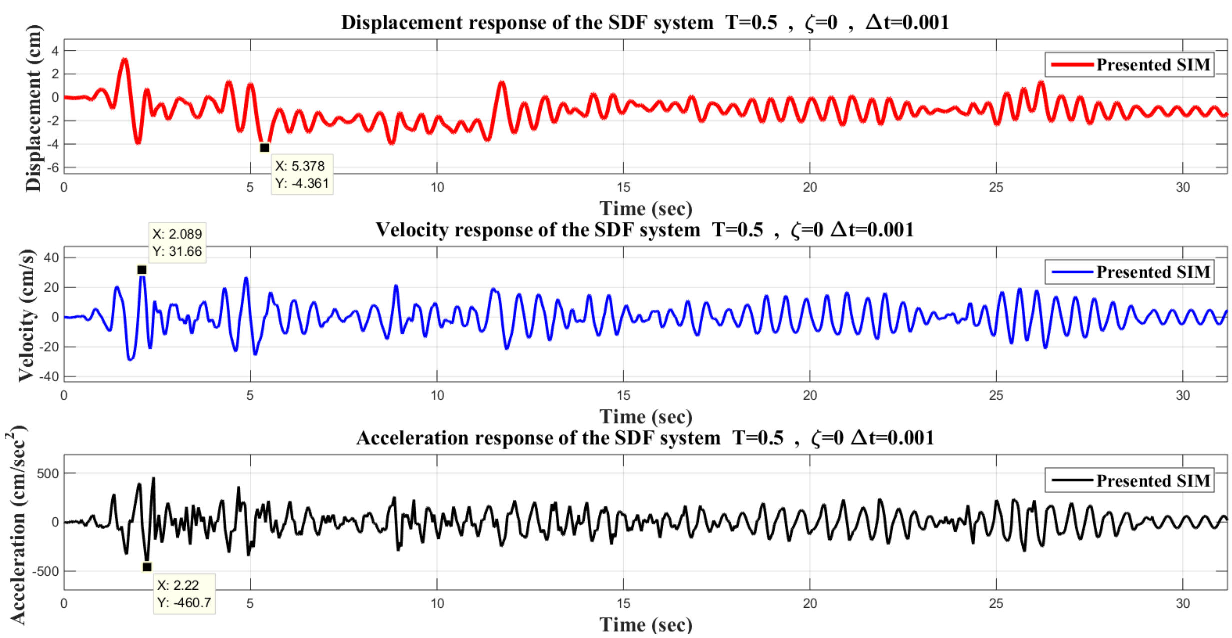


**Fig. 15.** Responses obtained from SIM for the nonlinear SDOF system in example III (Fine mesh).

**Table 6.** Peak response values and run times of nonlinear analyses in example III with $\Delta t = 0.001$ sec.

| method / item | Nonlinear Newmark-$\beta$ method | Presented SIM |
|---|---|---|
| Max Disp. (cm) | 4.3611 | 4.3610 |
| Max Vel. (cm/sec) | 31.658 | 31.660 |
| Max Accel. (cm/sec$^2$) | 460.70 | 460.7 |
| Number of iteration | 1 | 2 |
| Run time (sec) | 0.0086 | 0.0088 |

## 4. Conclusions

Two numerical methods were proposed for dynamic analysis of structural systems: energy-based method (EBM) and simplified integration method (SIM). EBM uses basic kinematic relations and energy concept. It is presented only for linear damped systems. SIM is a more general method for linear and nonlinear analysis of vibration. It benefits from the function approximation theory and integration platform in the formulation.

Current version of EBM has some deficiencies: (1) EBM cannot be converged for undamped systems and those ones whose damping ratios are less than 1%. (2) Convergence speed of EBM is dissatisfying and it needs tens of iteration to give precise response.

SIM is a general method which has noticeable advantages as follows: (1) wide variety of vibration problems is covered. (2) Highly-exact solution is offered. (3) Any type of nonlinearities can be included. (4) Free vibration can also be dealt with, as well as forced vibration. (5) Complex earthquake excitation can be analyzed. (6) Vastly simple and straightforward calculation is required. (7) Computer programming is remarkably facilitated for nonlinear analysis. (8) Beginners with a little theoretical and technical background in this field can use this method.

The performance of both methods is carefully investigated through numerous.

examples. Detailed discussion on mathematical assessment of convergence and stability of the methods is avoided; however, convergence and stability of the solution are graphically verified through the general examples. In conclusion, presented SIM introduces a robust and fast analysis tool for nonlinear analysis of vibration in structural dynamics.

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